

## UNIT 2: SECOND ORDER DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

### *Specific Objectives:*

1. To acquire skills in solving some specific second order differential equations.
2. To apply relevant skills of forming and solving second order differential equations in some given physical situations.
3. To be able to interpret the solutions of second order differential equations.

Detailed Content	Time Ratio	Notes on Teaching
2.1 Classification of Types	2	<p>This is an extension of the first order differential equations. Here the emphasis is on the linear equations of the second order, i.e. equations of the type <math>\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)</math></p> <p>The main feature of this equation (i.e. it is linear in <math>y</math> and its derivatives while <math>p</math>, <math>q</math> and <math>f</math> are any given functions of <math>x</math>) should be clearly stated. Teachers should provide adequate examples to help students identify the various types of second order differential equations, namely, homogeneous linear equations (<math>f(x) = 0</math>), non-homogeneous linear equations (<math>f(x) \neq 0</math>) and non-linear equations (equations which cannot be written in the above form).</p> <p>At this stage, teachers may introduce examples like oscillation of a body hung on the bottom of a suspended spring, free falling of a body under the influence of a constant gravitational force and a resistance proportional to the speed, and pricing policy for optimum inventory level etc to indicate to students how second order differential equations arise in the real-life world.</p>
2.2 Principle of Superposition	2	<p>The principle can be introduced by using a concrete example. For example, students may be asked to verify that <math>y = x</math> and <math>y = x^2</math> are solutions of the equation <math>x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0</math>. Then, they are encouraged to go a step further to verify that <math>y = 3x + 4x^2</math> is also a solution. After trying a few examples in this way, teachers may guide students to prove the principle. However, for less able students, the formal proof can be omitted.</p> <p>Teachers are reminded that a formal knowledge of linear independence of solutions is not expected.</p> <p>The invalidity of the principle of superposition for non-homogeneous linear equations or non-linear equations can be</p>

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<p data-bbox="201 765 674 952"><b>2.3 Solution of Homogeneous Equations with Constant Coefficients</b></p> $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$	4	<p data-bbox="810 359 1938 667">introduced by the use of examples such as <math>\frac{d^2 y}{dx^2} + y = 1</math> in which <math>y = 1 + \sin x</math> and <math>y = 1 + \cos x</math> are solutions but <math>y = 3(1 + \sin x)</math> and <math>y = (1 + \sin x) + 2(1 + \cos x)</math> are not. Students should then be clear that the principle only holds for homogeneous linear differential equations.</p> <p data-bbox="810 765 1938 905">Here students are only expected to use the method of auxiliary equation (or characteristic equation). Other methods are unnecessary.</p> <p data-bbox="810 1041 1938 1676">Teachers should discuss with students separately the 3 cases arisen, i.e. when the auxiliary equation has 2 real and distinct roots, 2 real and equal roots, and 2 complex conjugate roots. For the last case, if the roots are <math>u \pm vi</math>, the standard solutions <math>y = (c_1 \cos vx + c_2 \sin vx)e^{ux}</math> can be obtained by the use of substitution <math>y = ze^{ux}</math>. Then the equation becomes <math>\frac{d^2 z}{dx^2} + v^2 z = 0</math>. Clearly, <math>z = \cos vx</math> and <math>z = \sin vx</math> are two distinct solutions. Hence, by the principle of superposition, the general solution is <math>z = c_1 \cos vx + c_2 \sin vx</math>. After putting back <math>z (= ye^{-ux})</math>, the result follows. For abler students, teachers may apply the identity <math>e^{i\theta} = \cos \theta + i \sin \theta</math> in the proof.</p> <p data-bbox="810 1724 1938 1863">This section is of great use for later work. Thus, more practice should be given to ensure that students master the technique. Real-life applications may be left to section 2.7.</p>
<p data-bbox="201 1970 674 2157"><b>2.4 Solution of Non-homogeneous Equations with Constant Coefficients</b></p> $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$ <p data-bbox="304 2418 674 2558">(a) Complementary function and particular integral</p>	6	<p data-bbox="863 1970 1938 2012">The following theorem should be introduced and clearly explained.</p> <p data-bbox="810 2059 1938 2249">The general solution of a non-homogeneous linear differential equation is the sum of a general solution of the reduced homogeneous equation (i.e. with <math>f(x)</math> setting zero) and an arbitrary particular solution of the non-homogeneous equation.</p> <p data-bbox="810 2297 1938 2537">Teachers should also emphasize that to avoid confusion, it is usual to call the general solution of the reduced equation the 'complementary function' and the particular solution of the non-homogeneous equation a 'particular integral'. Therefore, for non-homogeneous equations, we have,</p> <p data-bbox="892 2585 1921 2626" style="text-align: center;">general solution = complementary function + particular integral</p>

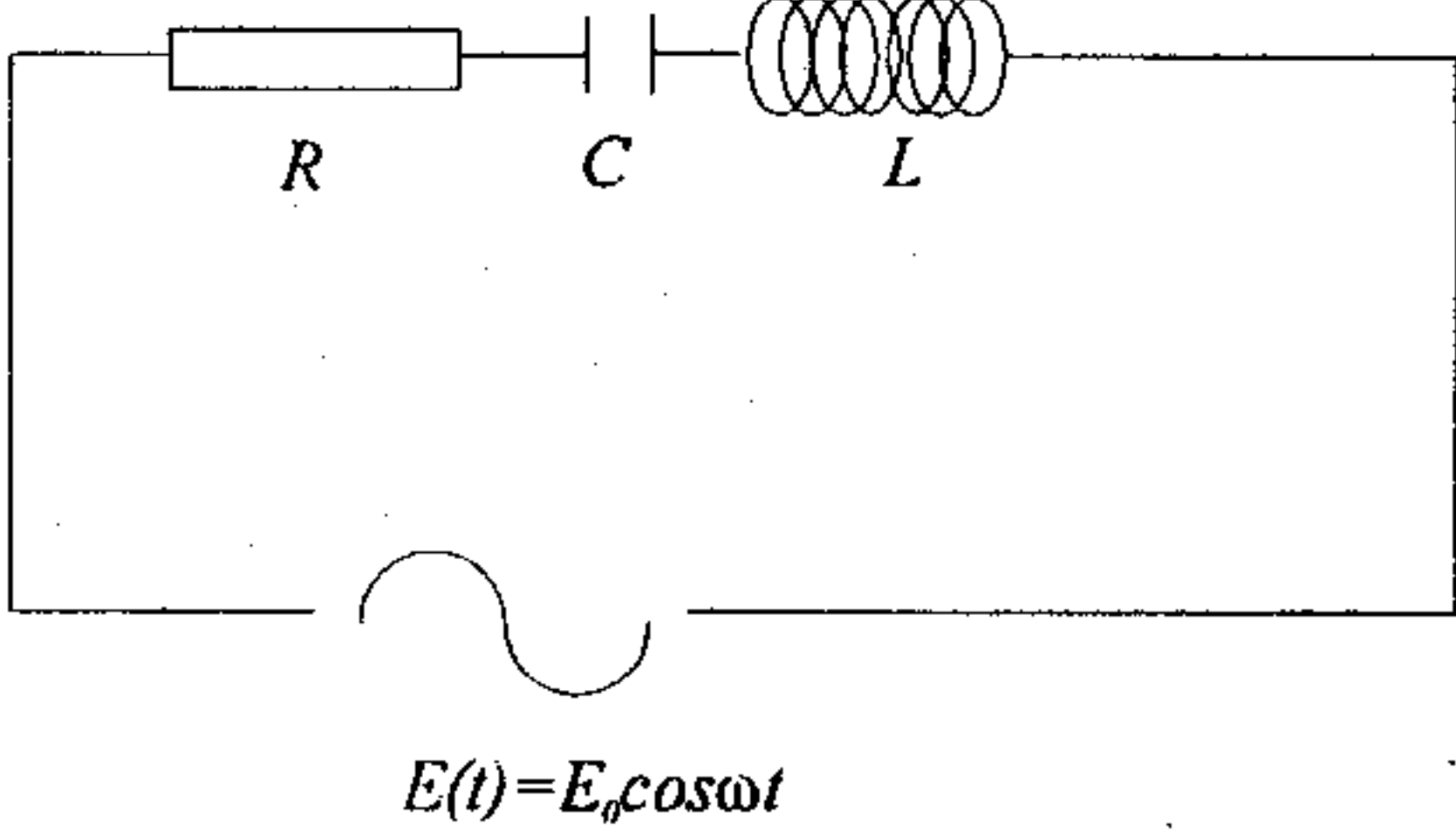
Detailed Content	Time Ratio	Notes on Teaching										
(b) Method of undetermined coefficients		<p>The proof of the above theorem can be left to students as an exercise.</p> <p>Only the method of undetermined coefficients is expected here. The method of inverse operator should be avoided.</p> <p>To help students memorize the possible trial forms of the particular integral <math>y_p(x)</math> (which is dependent on the form of the function <math>f(x)</math>), teachers may introduce the following table.</p> <table border="1" data-bbox="841 1006 1866 1471"> <thead> <tr> <th data-bbox="841 1006 1307 1096"><i>Form of <math>f(x)</math></i></th> <th data-bbox="1307 1006 1866 1096"><i>Trial form of <math>y_p(x)</math></i></th> </tr> </thead> <tbody> <tr> <td data-bbox="841 1096 1307 1186"><math>x^n</math></td> <td data-bbox="1307 1096 1866 1186"><math>a_0 + a_1x + a_2x^2 + \dots + a_nx^n</math> (o)</td> </tr> <tr> <td data-bbox="841 1186 1307 1276"><math>e^{px}</math></td> <td data-bbox="1307 1186 1866 1276"><math>Ae^{px}</math> (p)</td> </tr> <tr> <td data-bbox="841 1276 1307 1366"><math>c \cos px</math></td> <td data-bbox="1307 1276 1866 1366" rowspan="3">} <math>A \cos px + B \sin px</math> (ip)</td> </tr> <tr> <td data-bbox="841 1366 1307 1456"><math>c \sin px</math></td> </tr> <tr> <td data-bbox="841 1456 1307 1471"><math>c_1 \cos px + c_2 \sin px</math></td> </tr> </tbody> </table> <p>However, teachers should remind students that if the number listed in the brackets is a root of multiplicity <math>k</math> (<math>k = 1</math> or <math>2</math>) of the auxiliary equation of the reduced homogeneous equation, then the trial form of <math>y_p(x)</math> is <math>x^k</math> times the above form.</p> <p>Examples such as the following can be used to illustrate the rationale.</p> <p><i>Example 1</i></p> <p>For the equation <math>\frac{d^2y}{dx^2} + 9y = \cos 2x</math>, the roots of the auxiliary equation are <math>\pm 3i</math> and hence the complementary function is <math>y_c = c_1 \cos 3x + c_2 \sin 3x</math>. Since <math>2i</math> is not a root of the auxiliary equation, the particular integral to be tried is <math>y_p = A \cos 2x + B \sin 2x</math>. However, if the equation is <math>\frac{d^2y}{dx^2} + 9y = \cos 3x</math>, the complementary function remains the same but since <math>3i</math> is a root of the auxiliary equation, the particular integral to be tried should be <math>y_p = x(A \cos 3x + B \sin 3x)</math>.</p>	<i>Form of <math>f(x)</math></i>	<i>Trial form of <math>y_p(x)</math></i>	$x^n$	$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ (o)	$e^{px}$	$Ae^{px}$ (p)	$c \cos px$	} $A \cos px + B \sin px$ (ip)	$c \sin px$	$c_1 \cos px + c_2 \sin px$
<i>Form of <math>f(x)</math></i>	<i>Trial form of <math>y_p(x)</math></i>											
$x^n$	$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ (o)											
$e^{px}$	$Ae^{px}$ (p)											
$c \cos px$	} $A \cos px + B \sin px$ (ip)											
$c \sin px$												
$c_1 \cos px + c_2 \sin px$												

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		<p>In either case, students should be clear that they have to substitute <math>y_p</math> into the equation to calculate the values of <math>A</math> and <math>B</math>.</p> <p><i>Example 2</i></p> <p>For the equation <math>\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2</math>, since 0 is one of the roots of the auxiliary equation (whose roots are 0 and -2), the particular integral to be tried is <math>y_p = x(A + Bx + Cx^2)</math>. Again, the values of <math>A</math>, <math>B</math> and <math>C</math> can be calculated by putting back <math>y_p</math> into the equation. Students may have difficulty when dealing with equations where <math>f(x)</math> is a linear combination of functions in the first column of the above table. Therefore, teachers should provide examples such as that shown below to illustrate that the trial form of <math>y_p(x)</math> is the linear combination of the functions of the corresponding lines.</p> <p><i>Example 3</i></p> <p>For the equation <math>\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}</math>, roots of the auxiliary equation are 1 and 2. Since 0 and 3 are not roots of the auxiliary equation, the trial form of <math>y_p(x)</math> is</p> $y_p = A_0 + A_1x + Be^{3x}.$ <p>Teachers should indicate that <math>A_0 + A_1x</math> is the particular solution for <math>4x</math> while <math>Be^{3x}</math> is that for <math>e^{3x}</math>.</p> <p>However, if the equation is <math>\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^x</math>, then since 1 (coefficient of <math>x</math> in the index of <math>e^x</math>) is a root of the auxiliary equation, the trial form of <math>y_p(x)</math> is <math>y_p = \underbrace{A_0 + A_1x}_{\text{for } 4x} + \underbrace{Bxe^x}_{\text{for } e^x}</math>.</p> <p>Adequate practice of initial or boundary value problems should be provided in order to ensure that students are familiar with practical procedures required in solving real-life problems.</p>

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2.5 Reduction of Equations to Second Order Differential Equations with Constant Coefficients	3	<p>Students are expected to be able to make use of a given substitution to reduce a differential equation to one of the familiar types. For example,</p> <p>1. <math>x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0</math> can be reduced to <math>\frac{d^2 y}{dz^2} + y = 0</math> by substituting <math>x = e^z</math>. In this example, teachers should remind students that <math>\frac{d^2 y}{dz^2} \neq \frac{d^2 y}{dx^2} \cdot \frac{d^2 x}{dz^2}</math>.</p> <p>2. <math>x^2 \frac{d^2 y}{dx^2} + 2x(x+2) \frac{dy}{dx} + 2(x+1)^2 y = e^{-x}</math> can be reduced to <math>\frac{d^2 z}{dx^2} + 2 \frac{dz}{dx} + 2z = e^{-x}</math> by making use of the substitution <math>y = \frac{z}{x^2}</math>.</p> <p>In all these types of examples, teachers should emphasize to or revise (if necessary) with students the formulae <math>\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}</math> and <math>\frac{d^2 y}{dx^2} = \frac{dy}{dz} \cdot \frac{d^2 z}{dx^2} + \frac{dz}{dx} \cdot \frac{d}{dx} \left[ \frac{dy}{dz} \right] = \frac{dy}{dz} \cdot \frac{d^2 z}{dx^2} + \frac{dz}{dx} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}</math>.</p>
2.6 Systems of two First Order Differential Equations	3	<p>Only simple systems which may be reduced by elimination to a second order linear differential equation is expected. For example, the two equations <math>\frac{dy}{dt} - x = t</math> and <math>\frac{dx}{dt} + y = t^2</math> can be reduced to <math>\frac{d^2 y}{dt^2} + y = 1 + t^2</math> while the two equations <math>\frac{dx}{dt} = x - 3y</math> and <math>\frac{dy}{dt} = y - 3x</math> to <math>\frac{d^2 x}{dt^2} - 2 \frac{dx}{dt} - 8x = 0</math>.</p>
2.7 Applications in Practical Problems	8	<p>There are many real-life problems which can lead to second order differential equations. The following are some examples.</p> <p>1. Pricing policy for the production of goods</p> <p>The following shows one of the various models of a company's pricing policy on the goods produced.</p> $\frac{dP}{dt} = -k(L(t) - L_0)$

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		$\frac{dL}{dt} = Q(t) - S(t)$ $S(t) = 500 - 40P - 10\frac{dP}{dt}$ $Q(t) = 250 - 5P$ <p>where <math>P(t)</math> = price of goods</p> <p style="padding-left: 40px;"><math>S(t)</math> = forecasting sales</p> <p style="padding-left: 40px;"><math>L(t)</math> = inventory level</p> <p style="padding-left: 40px;"><math>Q(t)</math> = production level</p> <p style="padding-left: 40px;"><math>L_0</math> = optimum level</p> <p style="padding-left: 40px;"><math>k</math> = positive constant</p> <p>By differentiation, students should have no difficulty to get</p> $\frac{d^2P}{dt^2} = -k \frac{dL}{dt} = -k(Q(t) - S(t))$ $= -k(-250 + 35P + 10\frac{dP}{dt})$ <p>or <math>\frac{d^2P}{dt^2} + 10k\frac{dP}{dt} + 35kP = 250k</math></p> <p><b>2. Population growths of 2 countries</b></p> <p>Two countries <math>A</math> and <math>B</math> have the same natural growth rate of <math>\lambda\%</math> per annum. The emigration rate of <math>A</math> to <math>B</math> is <math>a\%</math> per annum while that of <math>B</math> to <math>A</math> is <math>b\%</math> per annum. Suppose the initial populations of <math>A</math> and <math>B</math> are <math>N_1</math> and <math>N_2</math> respectively. Then their populations after <math>t</math> years (denoted by <math>x(t)</math> and <math>y(t)</math> respectively) are given by <math>\frac{dx}{dt} = \frac{\lambda x}{100} - \frac{ax}{100} + \frac{by}{100}</math> and <math>\frac{dy}{dt} = \frac{\lambda y}{100} - \frac{by}{100} + \frac{ax}{100}</math> which can be easily reduced to a second order differential equation.</p>

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		<p data-bbox="856 368 1209 409">3. <i>Chemical Reaction</i></p> <p data-bbox="846 460 1934 804">In a chemical reaction there are present, at time <math>t</math>, <math>x</math> kg of substance <math>X</math> and <math>y</math> kg of substance <math>Y</math>, and initially there is 2 kg of <math>X</math> and 4 kg of <math>Y</math>. The variables <math>x</math> and <math>y</math> satisfy the equations <math>\frac{dx}{dt} = -x^2y</math>, <math>\frac{dy}{dt} = -xy^2</math>. Find <math>\frac{dy}{dx}</math> in terms of <math>x</math> and <math>y</math>, and express <math>y</math> in terms of <math>x</math>. Hence obtain a differential equation in <math>x</math> and <math>t</math> only, and so find an expression for <math>x</math> in terms of <math>t</math>.</p> <p data-bbox="856 943 1199 985">4. <i>Cooling of a body</i></p> <p data-bbox="846 1035 1934 1427">The temperature <math>y</math> degrees of a body, <math>t</math> minutes after being placed in a certain room, satisfies the differential equation <math>3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0</math>. By using the substitution <math>z = \frac{dy}{dt}</math>, or otherwise, find <math>y</math> in terms of <math>t</math>, given that <math>y = 60</math> when <math>t = 0</math> and <math>y = 35</math> when <math>t = 6 \ln 4</math>. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?</p> <p data-bbox="856 1567 1136 1608">5. <i>Charge losses</i></p> <p data-bbox="846 1659 1934 1955">An electrically charged body losses its charge, <math>Q</math> coulombs, at a rate <math>KQ</math> coulombs per second, where <math>k</math> is a constant. Write down a differential equation involving <math>Q</math> and <math>t</math>, where <math>t</math> seconds is the time since the discharge started. Solve the equation for <math>Q</math>, given that the initial charge was <math>Q_0</math> coulombs. If <math>Q_0 = 0.001</math>, and <math>Q = 0.0005</math> when <math>t = 10</math>, find the value of <math>Q</math> when <math>t = 20</math>.</p> <p data-bbox="856 2095 1157 2136">6. <i>Electric circuit</i></p> <p data-bbox="846 2187 1934 2484">If students have learnt the concepts of inductance and capacitance (say in the A-level Physics), examples of electric circuits may be introduced for their interest. One of the possible equations is <math>L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E_0\omega \cos\omega t</math> when <math>R</math>, <math>C</math> and <math>L</math> are connected in series.</p>

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	28	 <p>The diagram shows a rectangular circuit loop. On the top horizontal wire, three components are connected in series from left to right: a resistor labeled <math>R</math> (represented by a rectangle), a capacitor labeled <math>C</math> (represented by two parallel vertical lines), and an inductor labeled <math>L</math> (represented by a series of loops). On the bottom horizontal wire, there is an AC voltage source represented by a sine wave. Below the sine wave, the equation <math>E(t) = E_0 \cos \omega t</math> is written.</p>