**UNIT 11: Motion of a Rigid Body**

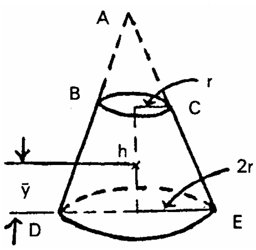
*Specific Objectives:*

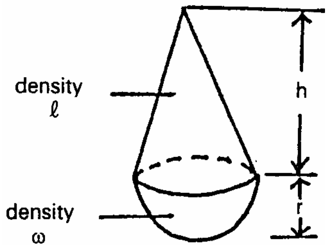
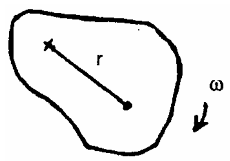
1. To find the centre of mass and the moment of inertia of a given rigid body.
2. To understand and use the law of conservation of angular momentum.
3. To solve dynamic problems involving rigid bodies.



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| **11.1 Centre of Mass**<br>(a) Introduction | 6 | As students may not be too familiar with the term 'rigid bodies', teachers may introduce the definition of centre of mass by first considering a finite number of particles.<br><br>Simple examples such as the one given below may help students to understand the formula.<br><br>$$\vec{r_c} = \frac{\sum m_i \vec{r_i}}{\sum m_i}$$<br><br>where $\vec{r_c}$ is the position vector of the centre of mass.<br><br>Example<br>Masses $m_1$, $m_2$, $m_3$ and $m_4$ are placed on the corners A. B, C and D respectively of a square of side a. Find the position of the centre of mass.<br><br>In this example, students should be able to calculate the position vector of the centre of mass of the system of particles with A as the origin. By using C as the origin, teachers can remind students that the centre of mass of a system of particles is independent of the reference points for the position vector. |
| (b) Centre of mass by integration | | When a body cannot be divided into a finite number of particles, it may be divided into a large number of very small parts called elements. The position of the centre of mass can be found by integration using the formulae |

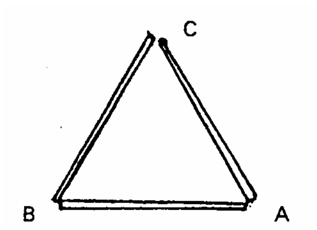
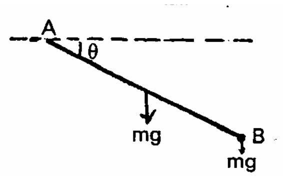


| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | $$\bar{x} = \frac{\int x \, \mathrm{d}m}{\int \mathrm{d}m} \quad \text{and} \quad \bar{y} = \frac{\int y \, \mathrm{d}m}{\int \mathrm{d}m}$$<br><br>Examples such as uniform rods, triangular lamina, solid and hollow hemisphere, circular arc etc should be discussed.<br><br>Various examples can be given to students to demonstrate the skills in finding the centre of mass of less familiar objects. For example, teachers may ask students to find the centre of mass of the uniform solid obtained by rotating about the x-axis, the area bounded by the x-axis, the line x = 2 and the parabola $y^2 = 4x$.<br><br>Students are expected to know that for rigid bodies involving solid of revolution (e.g. solid cone), the useful element is a disc, and that for solid involving surface of revolution (e.g. a hollow hemisphere), the useful element is a ring. |
| (c) Centre of mass of a composite body | | In finding the centre of mass of composite bodies, the relative mass of the different parts can be used.<br><br>*Example 1*<br>Find the centre of mass of a solid frustum.<br><br><br><br>The centre of mass of the frustum can be calculated by considering the relative masses of the fictitious cones ABC and ADE and the frustum BCDE. The results is<br><br>$$8\left(\frac{2h}{4}\right) = 7\bar{y} + (1)\left(h + \frac{h}{4}\right)$$ |

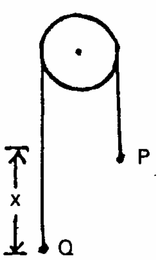
| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | *Example 2*<br>To find the centre of mass of a solid consisting of a right circular cone and a hemisphere.<br><br>In this example, students can be guided to find the possible ratio of h/r so that the body can rest in equilibrium with any part of the curved surface of the hemisphere in contact with a horizontal smooth plane. |
| **11.2 Moment of Inertia**<br>(a) Introduction | 6 | As an introduction, teachers may explain to students that the motion of a rigid body is quite different from that of a particle. The general motion may involve translation and rotation.<br>By considering the rotational motion of a rigid body about a fixed axis, most students should not have any difficulties in getting an expression for the kinetic energy of the body.<br><br>K.E. = $\frac{1}{2}\omega^2\left(\sum mr^2\right)$<br>Teachers may then introduce the term 'moment of inertia, I = Lmr1' and remind students that I is of great importance and occurs in all the problems involving the rotation of a rigid body. |
| (b) Moment of inertia by integration | | Teachers should guide students to find the moment of inertia of a finite number of particles and to extend the idea to find the moment of inertia of a rigid body about an axis. In the latter case, the technique of integration should be used. |



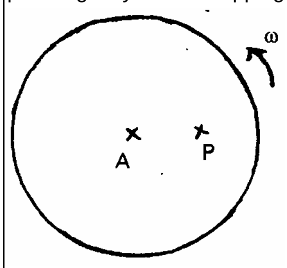
| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | Students should realize that the moment of inertia of a body depends on<br>(a) position of axis of rotation;<br>(b) distribution of mass about the axis.<br>Examples such as uniform rods, rectangular lamina, ring, disc, sphere etc. should be discussed. |
| (c) Parallel and perpendicular axes theorem | | At this stage, students should be familiar with the techniques of finding the moment of inertia of a rigid body from first principles. Teachers can then introduce parallel axis theorem and perpendicular axis theorem. These two theorems will help students in finding the moments of inertia of a body about other axes when the moments of inertia about certain standard axes are known. In this way a large amount of integration is avoided. Teachers should show students how to apply the two useful theorems in the calculation of the moment of inertia.<br>Examples such as finding the moment of inertia of a disc about a tangent and that of a solid cone about an axis through the vertex, perpendicular to the axis of symmetry may be discussed.<br>Teachers can remind students that<br>(a) perpendicular axis theorem can only be applied to rigid body in the form of a lamina;<br>(b) from parallel axis theorem, the moment of inertia about an axis through the centre of mass of a uniform body is less than that about any parallel axis by Md1 where M is the mass of the body and d is the distance between the parallel axes. |
| (d) Moment of inertia of a composite body | | Students are expected to know that the moment of inertia of a composite body can be obtained easily by adding the moments of inertia of individual parts together. Examples such as finding the moment of inertia of the following figure about the axis through A and perpendicular to the plane ABC can be discussed. |

| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | |  |
| **11.3 Moment of a Rigid Body about a Fixed Axis**<br>(a) Conservation of energy | | As the topic 'general motion of a rigid body' is a bit difficult for average students, it is desirable that teachers start to discuss first the motion of a rigid body about a fixed axis.<br><br>Students are expected to apply the law of conservation of energy in solving problems relating to rotation of a rigid body about a fixed axis.<br><br>*Example*<br>A uniform rod AB of mass m and length 2a is free to turn about a smooth horizontal axis about A. A particle of mass m is attached to the rod at B. The rod is released from rest with AB horizontal.<br><br>Teachers should remind students that they cannot treat the rod as a point mass.<br><br>Teachers can guide students to obtain the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ from the law of conservation of energy. |
| (b) Law of angular momentum | | The definition of angular momentum about a fixed axis can then be discussed with students. Students are expected to know the law of angular momentum.<br><br>$I\frac{d\omega}{dt} = I\ddot{\theta} = L$ where $L$ is the moment of the external forces acting on the body about the fixed axis. |



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | The above equation can be compared with Newton's second law $m\frac{dv}{dt} = F$ .<br><br>The following show some examples which can be discussed with students.<br><br>*Example 1*<br>The moment of inertia of a flywheel about its axis is 10 kg m$^2$. When it is rotating with an angular speed $\omega_0$ a constant torque of 20 Nm is applied to the flywheel for 3 seconds.<br>From $I\ddot{\theta} = L$ , students should have no problems to obtain $\ddot{\theta} = 2$ . Teachers can then guide students to obtain the equations $\dot{\theta} = \omega_0 + 2t$ , $\theta = \omega_0 t + \frac{1}{2}(2)t^2$ which can be compared with the equations obtained from constant linear acceleration:<br>$$v = u + at \, , \; s = ut + \frac{1}{2}at^2$$<br>*Example 2*<br>A cylinder of radius a and mass M is free to rotate about its axis which is horizontal. A light string hangs over the pulley and carries a particle P of mass m at one end and a particle Q of mass 2m at the other end. The string is rough enough not to slip on the pulley. The system starts from rest and the particle Q moves down a distance x at time t.<br><br>In this example, teachers can discuss with students why x = a$\theta$ and $\ddot{x} = a\ddot{\theta}$ and remind students that the tensions in the strings are not the same on the two sides of the pulley. |

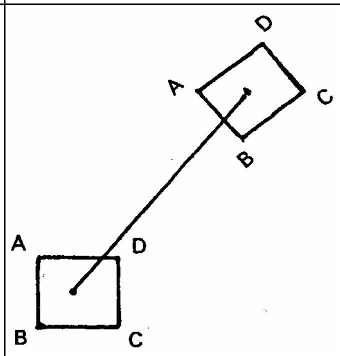
<table>
<tr><th>Detailed Content</th><th>Time Ratio</th><th>Notes on Teaching</th></tr>
<tr><td></td><td></td><td>
Students may be guided to set up the equations of motion for P, Q and the cylinder. Also, teachers can discuss with students the possibility of using energy conservation in this case.
The Law of conservation of angular momentum should be introduced and can be illustrated by considering simple examples such as the one given below:
Example<br>A uniform disc of mass m and radius a is rotating with constant angular velocity (j) in a horizontal plane about a vertical axis through its centre A. A particle P of mass 2m is placed gently without slipping on the disc at a point distant a/2 from A.



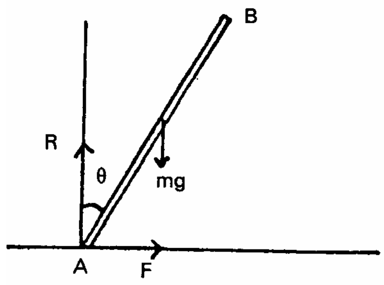
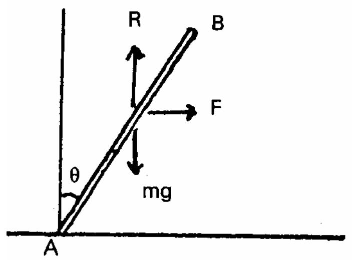

Teachers can explain to students that since there is no external torque acting on the system, angular momentum must be conserved.
Students can be guided to find the new angular velocity of the disc (which is the same as the particle).
</td></tr>
<tr><td>(c) Applications</td><td></td><td>Other examples such as the work done by a couple $\int L\, d\theta$, the impulse of the torque $\int L\, dt$, compound pendulum etc should also be discussed through worked examples.</td></tr>
<tr><td>11.4 General Motion of a Rigid Body<br>(a) Introduction</td><td>16</td><td>The general motion of a rigid body can be illustrated by diagrams.</td></tr>
</table>



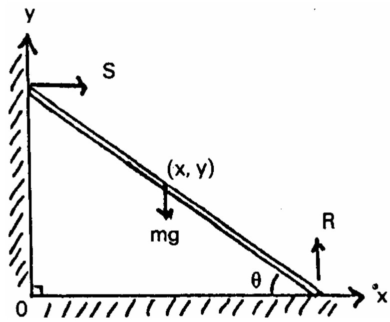
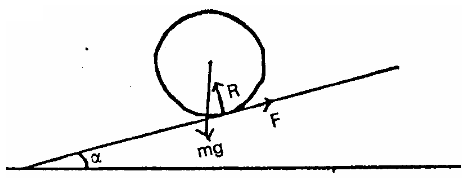
<table>
<tr><th>Detailed Content</th><th>Time Ratio</th><th>Notes on Teaching</th></tr>
<tr><td></td><td></td><td>

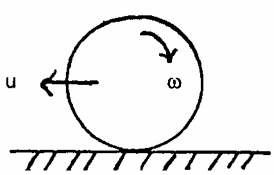
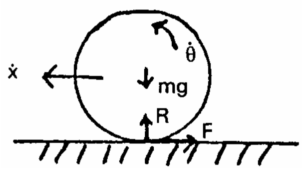


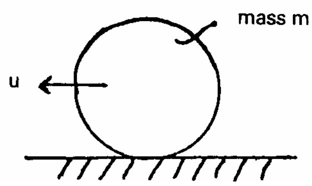

Students should know that the general motion consists of two parts:
(1) translation of the centre of mass
(2) rotation about the centre of mass
Teachers may derive the necessary equations required but emphasis should be placed on the application rather than derivation.
It is important for students to realize that the general motion of a rigid body can be analysed by considering independently<br>(a) the linear motion of the centre of mass<br>(b) the rotation about an axis through the centre of mass.
</td></tr>
<tr><td>(b) Equation of Motion</td><td></td><td>
Teachers can discuss with students the law of angular momentum which can be generalized as follows:
The rate of change of angular momentum of the body about the axis through the centre of mass is equal to the sum of moments of the external forces about the axis.
Examples should be introduced to demonstrate the techniques in solving problems relating to general motion.
</td></tr>
</table>

| Detailed Content | Time Ratio | Notes on Teaching |
| --- | --- | --- |
| | | *Example 1*<br>A uniform rod AB of length 2a and mass m is held vertically with one end resting on a horizontal plane which is rough enough to prevent slipping and is then released.<br><br>It is important for students to realize the fact that the motion of the centre of mass of a rigid body acted on by any forces, is the same as if the whole mass were collected at the centre of mass and all the forces were applied at that point.<br><br>Since the centre of mass is performing circular motion, students can be guided to obtain the equation of motion of centre of mass. By taking moment about A and considering the energy, $\dot{\theta}^2$ and $\ddot{\theta}$ can be obtained, thus F and R can be calculated.<br>Students may also be asked to write down the equation of motion by considering the forces along AB and perpendicular to AB. |



| Detailed Content | Time Ratio | Notes on Teaching |
| --- | --- | --- |
| | | *Example 2*<br>A uniform rod of length 2a slides down a vertical plane, its ends being in contact with two smooth planes, one horizontal and one vertical. The co-ordinates of the centre of mass is ($x$, $y$).<br><br>By considering the translational motion of the centre of mass and the rotational motion about the centre of mass, S and R can be obtained. |
| (c) Rolling and sliding | | Among the topics of the general motion of rigid bodies, 'pure rolling' and 'rolling with slipping' are two important ones. Below are some examples which can be discussed with students.<br>*Example 1*<br>A cylinder of radius a rolls down a rough inclined plane. Students should have no difficulties in getting the equations<br>$I\ddot{\theta} = Fa$ &<br>$m\ddot{x} = mg\sin\alpha - F$<br> |

| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | The velocity of the contact point A is $\dot{x} - a\dot{\theta}$ which may not be zero in the case of slipping.<br>In the case of rolling without slipping, students should know that $\dot{x} = a\dot{\theta}$.<br>*Example 2*<br><br>A cylinder with a backward angular velocity ω and a velocity u is placed on a rough horizontal plane. Assume it rotates an angle θ and moves a distance x at time t.<br><br>The velocity of the contact point A is $\dot{x} - a\dot{\theta}$<br>Teachers can investigate the motion of the cylinder eventually if<br>(1) $u > a\omega$ (2) $u = a\omega$ (3) $u < a\omega$<br>Teachers should emphasize to students that friction should not be assumed to be limiting until slipping has been established.<br>Other examples such as the motion of a solid, sphere rolling on a fixed sphere, the motion of a solid cylinder moving inside a hollow cylinder can also be discussed. |
| (d) General expression of the kinetic energy of a rigid body | | The general law of kinetic energy should be discussed. The kinetic energy of the body is made up of linear kinetic energy of a particle of mass M at the centre of mass and the rotational kinetic energy of the body about the axis through the centre of mass.<br>Examples such as the one given below can be discussed. |

| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | *Example*<br><br>A cylinder is rolling without slipping along a horizontal plane with speed u. Students should be able to write down the expression of the K. E. of the cylinder, i.e.<br>K.E. = $\frac{1}{2}I\left(\frac{u}{a}\right)^2 + \frac{1}{2}mu^2$ |
| | 34 | |