UNIT 12: First Order Differential Equations and Its Applications

Specific Objectives:

1. To acquire skills in solving some specific first order differential equations.
2. To apply relevant skills of forming and solving first order differential equations in some given physical situations.
3. To be able to interpret the solutions of first order differential equations.

Detailed Content	Time Ratio	Notes on Teaching
12.1 Basic Concepts and Ideas	3	Teachers may make use of simple examples like $\frac{dy}{dx}=x^2$ and $\frac{d^2y}{dx^2}=x^3$ to introduce the general concept of differential equations (equations containing differential coefficients) and ask students to find the solutions of examples so given. Students should have no problem as those examples can be solved by simple integration. (The solution of the first is given by integrating x^2 once while that of the second by integrating x^3 twice.) But, how about the equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$? For students' interest, teachers may introduce the term 'ordinary differential equation'. But since only one independent variable is considered in this topic area, this can be simply called differential equation if no ambiguity arises. Students are also expected to recognise the terms 'order', 'degree', 'linear' and 'non-linear' in differential equations. Examples should be given to clarify the various concepts. The meaning of a solution of a differential equation should be clearly explained. This can be done through examples. For example, $f(x)=e^{2x}$ is a solution of the differential equation $\frac{dy}{dx}-2y=0$ because $f'(x)-2f(x)=0$.

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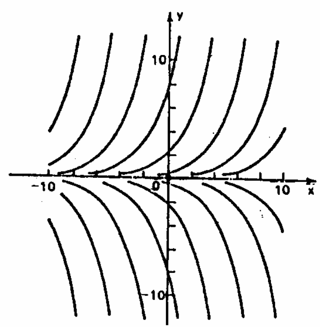
		' Arbitrary constants', 'general solution' and 'particular solution' are essential ideas in this topic area. In particular, students are expected to know that any solution of an nth order equation NOT containing n arbitrary constants cannot be a general solution and the solution that has satisfied certain specific conditions, known as initial or boundary conditions, is called a particular solution. These concepts can be further clarified by means of graphs. For example, the general solution $y=ce^{2x}$ of the equation $\frac{dy}{dx}-2y=0$ represents a family of curves as shown in the figure below while the particular solution $y=e^{2x}$ (or $y=2e^{2x}$ etc.) represents only one curve in the family.  Students are also expected to identify the number of arbitrary constants in a function. For example, the function given by $c_1e^{c_2+x}$ appears to contain two arbitrary constants, but in fact it contains only one as we can write $c_1e^{c_2+x}=(c_1e^{c_2})e^x=ce^x$ where $c_1e^{c_2}$ is replaced by the single arbitrary constant *c*. For the abler students, teachers can also discuss singular solution with them. For example, $y=cx+\frac{1}{c}$ is the general solution of $y\frac{dy}{dx}=x\left(\frac{dy}{dx}\right)^2+1$ while $y^2=4x$ is a singular solution.

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12.2 Formation of Differential Equations	2	The emphasis here is on the formation of differential equations from physical situations. The solving of the resulting equations is left to the later sections. Teachers may provide some illustrative examples to indicate to students how this can be done. A typical example is the problem of growth in which the rate of change of population of a certain species, $\frac{\mathrm{d}P}{\mathrm{d}t}$ at any time t is proportional to the value of P at that instant. Students should be able to write down the relation $\frac{\mathrm{d}P}{\mathrm{d}t} \propto \mathrm{P}$ or $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ where k, known as the growth constant, is a positive constant.
12.3 Solution of Equations with Variables Separable	4	Students should be able to identify differential equations with variables separable and reduce them to the form $g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x$. Accordingly, students should have no problem to solve the equations by simple integration. Since in many real life applications, people are not so interested in the general solution of a given differential equation but only in the particular solution satisfying a given initial condition, teachers are advised to provide more initial value problems to their students. There are many physical problems which can lead to first order differential equations of variables separable type. The following are some of them. 1. *Population growth* The population of a given species is decreased at a constant rate of n people per annum by emigration. And the population due to birth and death is increased at a constant rate of λ% of the existing population per annum. If the initial population is N people, then the population x people after t years is given by $\frac{dx}{dt} = \frac{\lambda}{100}x - n$. 2. *Exponential decay* The rate of decay of a radioactive substance at time t is proportional to the mass $x(t)$ of the substance left at that time. Thus, $\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu x$ where μ is a positive constant.

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		3. *Law of cooling* The rate of change of temperature of a body is proportional to the difference between the temperature of the body and the temperature θ of the surrounding medium. Suppose T is the temperature of the body at time t, then $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - \theta)$ where $k < 0$. 4. *Diffusion* A porous pot containing a solution of a substance of concentration of x mgcm^{-3} is placed in a large vessel containing the same solution but of higher concentration c mgcm^{-3}. The concentration of the solution in the pot will increase due to diffusion. Assuming that c is constant, the rate of increase of concentration of the solution in the pot is proportional to the difference in concentration. Thus x satisfies the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = k(c - x)$ where k is a positive constant. 5. *Evaporation* A wet and porous substance loses its moisture at a rate proportional to the moisture content, $x(t)$. Thus, the equation is $\frac{\mathrm{d}x}{\mathrm{d}t} = -kx$ where k is a positive constant. 6. *Chemical reaction* If the temperature is kept constant, the velocity of a chemical reaction is proportional to the product of the concentration of the substances which are reacting. If x represents the amount of the substance formed in the reaction, then x must satisfy the equation $\frac{\mathrm{d}x}{\mathrm{d}t} = k(a - x)(b - x)$ where k is a positive constant, while a and b are the original amounts of the two reacting substances respectively.

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		7. *Logistic law of population growth* An improved population model makes use of the differential equation $\frac{dN}{dt} = aN - bN^2$ in which aN is the birth rate, $-bN^2$ is the death rate and N is the population at that instant. Here a and b are positive constants. 8. *Spread of disease* Here the equation considered is $\frac{dN}{dt} = kN(P - N)$ where N is the population that is infected at time t, P is the total population which is susceptible to infection, and the rate of change of N is assumed to be proportional to the product of N and $P - N$. In the above examples, teachers, apart from guiding students to set up and solve the differential equations, should emphasize on the interpretation of the solutions.
12.4 Solution of Linear Differential Equations $\frac{dy}{dx} + p(x)y = q(x)$	4	For equations which can be reduced to the linear form $\frac{dy}{dx} + p(x)y = q(x)$, the solutions can be obtained by the use of integrating factor. Teachers should emphasize that as the equation is of the first order the solution contains only one arbitrary constant. Therefore, the integrating factor $e^{\int p(x)\,dx}$ should contain no constant of integration as any primitive of $p(x)$ will serve the purpose. By experience, students may easily forget the form of the integrating factor and some even take $e^{\int p(x)\,dx + c}$ as the solution of the given equation. Therefore, more practice should be given to ensure that students master the technique. Physical examples which give rise to first order linear differential equations are recommended. The following are two examples. 1. *Mixture* . A tank contains 100 L of solution in which 10 kg of chemical is dissolved. Solution containing 2 kg of the chemical per litre flows into the tank at 5 Lmin^{-1}. The mixture is well-stirred and drawn off at 4 Lmin^{-1}. If x kg is the mass of the chemical in the tank, then x satisfies the differential equation $\frac{dx}{dt} + \frac{4x}{100+t} = 10$ with initial condition $x = 10$ at $t = 0$.

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		2. *Chain reaction* Radioactive element X decays into radioactive element Y which in turns decays into element Z. The sum of the masses of X, Y and Z at time t (denoted by x, y and z respectively) is constant. Suppose the respective rate of decay is proportional to the corresponding mass at time t and $x = M$, $y = z = 0$ at $t = 0$. Then the relevant differential equations are $\frac{dx}{dt} = -k_1 x$ and $\frac{dy}{dt} = k_1 x - k_2 y$. The first equation gives $x = Me^{-k_1 t}$, while the second one becomes $\frac{dy}{dt} + k_2 y = k_1 M e^{-k^1 t}$ by substitution.
12.5 Solution of Equations Reducible to Variables Separable Type or Linear Type	4	Students are expected to be able to reduce a differential equation to one of the above types by using a given substitution. For example, the non-linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$, n ≠ 0, n ≠ 1, known as Bernoulli's equation, can be reduced to a linear equation by substituting $u = y^{1-n}$. Also, the differential equation of the form $\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x)$, known as Riccati equation, can be reduced to a first order linear differential equation by using the substitution $y = Y + \frac{1}{u}$ if Y is a known solution (i.e. a particular solution Y is known).
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