**UNIT 13: Second Order Differential Equations and Its Applications**

*Specific Objectives:*

1. To acquire skills in solving some specific second order differential equations.
2. To apply relevant skills of forming and solving second order differential equations in some given physical situations.
3. To be able to interpret the solutions of first order differential equations.

| Detailed Content | Time Ratio | Notes on Teaching |
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| **13.1 Classification of Types** | 2 | This is an extension of the first order differential equations. Here the emphasis is on the linear equations of the second order, i.e. equations of the type $$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=f(x)$$ The main feature of this equation (i.e. it is linear in *y* and its derivatives while *p*, *q* and *f* are any given functions of *x*) should be clearly stated. Teachers should provide adequate examples to help students identify the various types of second order differential equations, namely, homogeneous linear equations ($f(x)$ = 0), non-homogeneous linear equations ($f(x) \neq 0$) and non-linear equations (equations which cannot be written in the above form).<br><br>At this stage, teachers may introduce examples like oscillation of a body hung on the bottom of a suspended spring, free falling of a body under the influence of a constant gravitational force and a resistance proportional to the speed, and pricing policy for optimum inventory level etc to indicate to students how second order differential equations arise in the real-life world. |
| **13.2 Principle of Superposition** | 2 | The principle can be introduced by using a concrete example. For example, students may be asked to verify that $y = x$ and $y = x^2$ are solutions of the equation $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$. Then, they are encouraged to go a step further to verify that $y=3x+4x^2$ is also a solution. After trying a few examples in this way, teachers may guide students to prove the principle. However, for less able students, the formal proof can be omitted.<br><br>Teachers are reminded that a formal knowledge of linear independence of solutions is not expected. |



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| | | The invalidity of the principle of superposition for non-homogeneous linear equations or non-linear equations can be introduced by the use of examples such as $\frac{d^2y}{dx^2}+y=1$ in which $y=1+\sin x$ and $y=1+\cos x$ are solutions but $y=3(1+\sin x)$ and $y=(1+\sin x)+2(1+\cos x)$ are not. Students should then be clear that the principle only holds for homogeneous linear differential equations. |
| **13.3 Solution of Homogeneous Equations with Constant Coefficients**<br>$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ | 4 | Here students are only expected to use the method of auxiliary equation (or characteristic equation). Other methods are unnecessary.<br><br>Teachers should discuss with students separately the 3 cases arisen, i.e. when the auxiliary equation has 2 real and distinct roots, 2 real and equal roots, and 2 complex conjugate roots. For the last case, if the roots are $u \pm vi$, the standard solutions $y=(c_1\cos vx+c_2\sin vx)e^{ux}$ can be obtained by the use of substitution $y = ze^{ux}$. Then the equation becomes $\frac{d^2z}{dx^2}+v^2z=0$. Clearly, $z=\cos vx$ and $z=\sin vx$ are two distinct solutions. Hence, by the principle of superposition, the general solution is $z=c_1\cos vx+c_2\sin vx$. After putting back $z$ (= $ye^{-ux}$). the result follows. For abler students, teachers may apply the identity $e^{i\theta}=\cos\theta+i\sin\theta$ in the proof.<br><br>This section is of great use for later work. Thus, more practice should be given to ensure that students master the technique. Real-life applications may be left to section 13.7. |
| **13.4 Solution of Non-homogeneous Equations with Constant Coefficients**<br>$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$<br>(a) Complementary function and particular integral | 6 | The following theorem should be introduced and clearly explained.<br><br>The general solution of a non-homogeneous linear differential equation is the sum of a general solution of the reduced homogeneous equation (i.e. with $f(x)$ setting zero) and an arbitrary particular solution of the non-homogeneous equation. |

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| | | Teachers should also emphasize that to avoid confusion, it is usual to call the general solution of the reduced equation the ‘complementary function’ and the particular solution of the non-homogeneous equation a ‘particular integral’. Therefore, for non-homogeneous equations, we have,<br>general solution = complementary function + particular integral<br>The proof of the above theorem can be left to students as an exercise. |
| (b) Method of undetermined coefficients | 2 | Only the method of undetermined coefficients is expected here. The method of inverse operator should be avoided.<br>To help students memorize the possible trial forms of the particular integral y (x) (which is dependent on the form of the function f(x), teachers may introduce the following table. |

| Form of $f(x)$ | Trial form of $y_p(x)$ | |
|---|---|---|
| $x^n$ | $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ | (0) |
| $e^{px}$ | $Ae^{px}$ | (p) |
| $c \cos px$<br>$c \sin px$<br>$c_1 \cos px + c_2 \sin px$ | $A \cos px + B \sin px$ | (ip) |

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| | | However, teachers should remind students that if the number listed in the brackets is a root of multiplicity *k* (*k* = 1 or 2) of the auxiliary equation of the reduced homogeneous equation, then the trial form of $y_p(x)$ is $xk$ times the above form.<br>Examples such as the following can be used to illustrate the rationale.<br>*Example 1*<br>For the equation $\frac{d^2y}{dx^2} + 9y = \cos 2x$ , the roots of the auxiliary equation are $\pm 3i$ and hence the complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$ . Since $2i$ is not a root of the auxiliary equation, the particular integral to be tried is $y_p = A \cos 2x + B \sin 2x$ . |



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| | | However, if the equation is $\frac{d^2y}{dx^2} + 9y = \cos 3x$ , the complementary function remains the same but since $3i$ is a root of the auxiliary equation, the particular integral to be tried should be $y_p = x(a \cos 3x + B \sin 3x)$ .<br>In either case, students should be clear that they have to substitute $y_p(x)$ into the equation to calculate the values of *A* and *B*.<br>*Example 2*<br>For the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$ , since 0 is one of the roots of the auxiliary equation (whose roots are 0 and −2), the particular integral to be tried is $y_p = x(a + Bx + Cx^2)$ .<br>Again, the values of *A*, *B* and *C* can be calculated by putting back $y_p(x)$ into the equation.<br>Students may have difficulty when dealing with equations where *f*(*x*) is a linear combination of functions in the first column of the above table. Therefore, teachers should provide examples such as that shown below to illustrate that the trial form of $y_p(x)$ is the linear combination of the functions of the corresponding lines.<br>*Example*<br>For the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$ , roots of the auxiliary equation are 1 and 2. Since 0 and 3 are not roots of the auxiliary equation, the trial form of $y_p(x)$ is $y_p = A_0 + A_1 x + Be^{3x}$ .<br>Teachers should indicate that $A_0 + A_1 x$ is the particular solution for $4x$ while $Be^{3x}$ is that for $e^{3x}$ .<br>However, if the equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$ , then since 1 (coefficient of *x* in the index of $e^x$) is a root of the auxiliary equation, the trial form of $y_p(x)$ is<br>$y_p = \underbrace{A_0 + A_1 x}_{\text{for } 4x} + \underbrace{Bxe^x}_{\text{for } e^x}$ |

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| | | | Adequate practice of initial or boundary value problems should be provided in order to ensure that students are familiar with practical procedures required in solving real-life problems. |
| **13.5** | **Reduction of Equations to Second Order Differential Equations with Constant Coefficients** | 3 | Students are expected to be able to make use of a given substitution to reduce a differential equation to one of the familiar types. For example,<br>1. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$ can be reduced to $\dfrac{d^2y}{dz^2}+y=0$ by substituting $x=e^z$. In this example, teachers should remind students that $\dfrac{d^2y}{dz^2}\neq\dfrac{d^2y}{dx^2}\cdot\dfrac{d^2x}{dz^2}$.<br>2. $x^2\dfrac{d^2y}{dx^2}+2x(x+2)\dfrac{dy}{dx}+2(x+1)^2y=e^{-x}$ can be reduced to $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2z=e^{-x}$ by making use of the substitution $y=\dfrac{z}{x^2}$.<br>In all these types of examples, teachers should emphasize to or revise (if necessary) with students the formulae $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$ and $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dz}\cdot\dfrac{d^2z}{dx^2}+\dfrac{dz}{dx}\dfrac{d}{dx}\left[\dfrac{dy}{dz}\right]$ $=\dfrac{dy}{dz}\cdot\dfrac{d^2z}{dx^2}+\dfrac{dz}{dx}\cdot\dfrac{d^2y}{dz^2}\cdot\dfrac{dz}{dx}$. |
| **13.6** | **Systems of two First Order Differential Equations** | 3 | Only simple systems which may be reduced by elimination to a second order linear differential equation is expected. For example, the two equations $\dfrac{dy}{dt}-x=t$ and $\dfrac{dx}{dt}+y=t^2$ can be reduced to $\dfrac{d^2y}{dx^2}+y=1+t^2$ while the two equations $\dfrac{dx}{dt}=x-3y$ and $\dfrac{dy}{dt}=y-3x$ to $\dfrac{d^2x}{dt^2}-2\dfrac{dx}{dt}-8x=0$. |
| **13.7** | **Applications in Practical Problems** | 8 | There are many real-life problems which can lead to second order differential equations. Physical interpretation of the solution to these problems should also be discussed. The following are some examples. |



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| | | 1. *Oscillation*<br>Teachers may discuss with students the phenomena of simple harmonic motion, damped harmonic motion and forced oscillation. Teachers should guide students to set up the relevant equations and interpret the solutions. In fact. teachers may relate this topic with Unit 9 (Simple Harmonic Motion).<br>2. *Mechanical problem*<br>Many examples in mechanics may lead to a second order differential equation. For example, a body falls freely under a constant gravitational force and a resistance proportional to its speed. Its equation of motion is<br>$m\dfrac{d^2x}{dt^2}=mg-k\dfrac{dx}{dt}$<br>Again, students are expected to derive the equation and interpret the solution.<br>3. Pricing policy for the production of goods<br>The following shows one of the various models of a company's pricing policy on the goods produced.<br>$\dfrac{dP}{dt}=-k\left(L(t)-L_0\right)$<br>$\dfrac{dL}{dt}=Q(t)-S(t)$<br>$S(t)=500-40P-10\dfrac{dP}{dt}$<br>$Q(t)=250-5P$<br>where $P(t)$ = price of goods<br>$S(t)$ = forecasting sales<br>$Q(t)$ = production level<br>$L(t)$ = inventory level<br>$L_0$ = optimum level<br>$k$ = positive constant |

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By differentiation, students should have no difficulty to get

$$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -k\frac{\mathrm{d}L}{\mathrm{d}t} = -k\big(Q(t) - S(t)\big)$$
$$= -k(-250 + 35P + 10\frac{\mathrm{d}P}{\mathrm{d}t})$$

or $$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} + 10k\frac{\mathrm{d}P}{\mathrm{d}t} + 35kP = 250k$$

4. *Population growths of 2 countries*

Two countries A and B have the same natural growth rate of $\lambda$% per annum. The emigration rate of A to B is $a$% per annum while that of B to A is $b$% per annum.

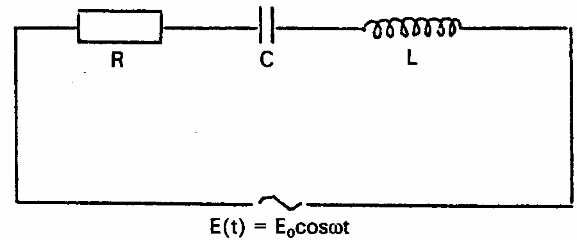
Suppose the initial populations of A and B are $N_1$ and $N_2$ respectively. Then their populations after t years (denoted by x(t) and y(t) respectively) are given by

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\lambda x}{100} - \frac{ax}{100} + \frac{by}{100}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\lambda y}{100} - \frac{by}{100} + \frac{ax}{100}$ which can be easily reduced to a second order differential equation.

5. *Electric circuit*

If students have learnt the concepts of inductance and capacitance (say in the A-level Physics), examples of electric circuits may be introduced for their interest.

One of the possible equations is $L\frac{\mathrm{d}^2\ell}{\mathrm{d}t^2} + R\frac{d\ell}{dt} + \frac{1}{C}\ell = E_0\omega\cos\omega t$ when $R$, $C$ and $L$ are connected in series.

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