UNIT 15: Approximation

Specific Objectives:

1. To learn the idea of approximation and the treatment of error.
2. To learn Taylor's series expansion.
3. To apply Taylor's series expansion to approximate functions and estimate the resulting errors.

Detailed Content	Time Ratio	Notes on Teaching
15.1 Treatment of Errors: their Estimation and Algebraic Manipulation	6	Analysis of the error in a numerical result is basic to any computation, whether done manually, with a calculator or a computer. Input values are seldom exact since they are often based on experiments or estimates, and the numerical processes themselves introduce errors of various types. In order to know how well the numerical results are, students are expected to be able to carry out simple error analysis in a calculation.
(a) Three basic types of errors		Students are expected to know that there are three basic types of errors in a numerical computation: inherent error, truncation error and round-off error.
(i) Inherent error		Inherent errors are errors in the values of data caused by uncertainty in measurement or by the necessarily approximating nature of representing in some finite number of digits a number that cannot be represented exactly in the number of digits available.
(ii) Truncation error		Truncation errors and round-off errors both refer to errors that are introduced by numerical procedures when the data are operated upon. The error introduced by truncating an infinite mathematical process is called truncation error. In numerical methods in this course, many of the procedures studied are infinite (in the sense that to obtain an exact solution would require an infinite number of iterations), so the subject of truncation error assumes major importance.
(iii) Round-off error		When a calculator or computer is used to perform real number calculations, round-off error occurs. This arises because the arithmetic performed in a machine involves numbers with only a finite number of digits, with the result that many calculations are performed with approximate representation of the actual numbers.

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		To illustrate the idea of rounding, teachers can use decimal machine numbers, represented in the normalized form $\pm 0.d_1d_2d_3\ldots d_k \times 10^n$, $1 \le d_1 \le 9$, $0 \le d_i \le 9$ for i = 2, 3, 4, ……, k. The idea of rounding up and rounding down should also be clearly explained.
(b) Absolute and relative error		It is preferable at this stage to revise with students the concept of absolute and relative error, which they have learned in S.3. Teachers should point out that each of the three types of errors in (a) can be expressed in absolute or relative form.
(c) Estimation of errors		A physical measurement may be given to a number of digits with the limits on inherent errors given such as in 2.3 ± 0.1 cm or 2.3 cm (2 sig. fig.) or without any qualification on the significance of the digits such as in 2.3 cm. In the latter case, it is often assumed that it is accurate to within half a unit in the last place i.e. 0.05 cm. As regards truncation errors, teachers should introduce them in the context of the procedures studied in the relevant sections of this unit. It is useful for teachers to guide students to derive a bound for the relative error using k-digit rounding arithmetic which is $0.5 \times 10^{-k+1}$.
(d) Combining errors		The major concern here is the question of how an error at one point propagates, that is, whether its effect becomes greater or smaller as subsequent operations are carried out. Students are expected to be familiar with the following facts. 1. Addition and subtraction If $S = a + b$ then $\max\lvert\Delta S\rvert = \lvert\Delta a\rvert + \lvert\Delta b\rvert$. If $K = a - b$ then $\max\lvert\Delta K\rvert = \lvert\Delta a\rvert + \lvert\Delta b\rvert$. 2. Multiplication and division If $P = ab$ then $\max\left\lvert\frac{\Delta P}{P}\right\rvert = \left\lvert\frac{\Delta a}{a}\right\rvert + \left\lvert\frac{\Delta b}{a}\right\rvert$. If $Q = \frac{a}{b}$ then $\max\left\lvert\frac{\Delta Q}{Q}\right\rvert = \left\lvert\frac{\Delta a}{a}\right\rvert + \left\lvert\frac{\Delta b}{a}\right\rvert$.

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		3. Exponentiation If $F = a^k$ then $\max\left\|\frac{\Delta F}{F}\right\| = k\left\|\frac{\Delta a}{a}\right\|$. The following examples are relevant. *Example* 1 Two masses are measured to be (100.0 ± 0.4) g and (94.0 ± 0.3) g. Calculate the maximum absolute error in the sum and difference in the masses. *Example* 2 The time period T of a simple pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$, where ℓ is the length of the pendulum and g is the acceleration of free fall due to gravity. A pendulum of length 0.600 m is used to determine the value of g. The value of T was found to be 1.55 s. Calculate the maximum percentage error in g.
15.2 Approximation of Functional Values using Taylor's Expansion	6	
(a) Taylor's series expansion of a function		Students may be motivated to find a polynomial $p(x)$ which has, for a single argument x_0, the values of the polynomial and its derivatives matching those of a function $f(x)$ i.e. $p^{(i)}(x_0) = f^{(i)}(x_0)$, i = 0, 1, 2, ..., n. By writing $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots\ldots + a_n(x - x_0)^n$ teachers may guide students to arrive at the Taylor's series expansion of $f(x)$, $p(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$. This idea may then be extended to the representation of a function by an infinite Taylor's series $f(x) = f(x_0) + f^{(1)}(x_0)\frac{(x - x_0)}{1!} + f^{(2)}(x_0)\frac{(x - x_0)^2}{2!} + \ldots\ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots\ldots$

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		The following may then be taken as examples for illustration. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots$ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$ At this stage it is worthwhile to mention Macclaurin series as a special case of Taylor's series. For the abler students, teachers may discuss with them the region of convergence for a Taylor's series as follows. Taking x_0 = 1 in the expansion of ln x, students should get $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots\ldots$ Putting x = 1 and 2, they would get easily that ln 1 = 0 and $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$ Then teachers may ask them to substitute x = 0 and 3 into the expansion and then lead them to discover that they are not convergent. Afterwards, students may be asked to evaluate some functions at some arguments within the region of convergence. It is preferable to ask students to expand a function in Taylor's series up to the fourth derivative only and do the evaluation correspondingly. Functions such as $y = (1 + x)^2 \ln(1 + x)$ are typical examples.
(b) Error estimation		Students are expected to recall the term truncation error as occurs in approximating a function using Lagrange Interpolating Polynomial. The remainder, R_n in Lagrangian form, could be shown or derived (for abler students only) to be $R_n = f^{(n)}(\varepsilon)\frac{(x - x_0)^n}{n!}$ where ε lies between x_0 and x.

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		Students should have no difficulty in seeing that the maximum error by truncating a series is given by $$\lvert R_n \rvert \le \left\lvert M \frac{(x-x_0)^n}{n!} \right\rvert$$ where $M = \max \lvert f^{(n)}(\varepsilon) \rvert$ for every ε between x_0 and x. Given this error term, students may be asked to do examples like estimating the maximum error committed if 3 terms in the series expansion of $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$ are used to evaluate $\cos\frac{\pi}{3}$, and finding the number of terms of the Taylor's series expansion of $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$ to evaluate e so that the maximum error is less than 0.0001.
	12	