UNIT 16: Numerical Integration

Specific Objectives:

1. To learn Trapezoidal rule and Simpson's rule.

2. To apply Trapezoidal rule and Simpson's rule in numerical integration and estimate their errors.

Detailed Content	Time Ratio	Notes on Teaching
16.1 Numerical Integration	1	The importance of numerical integration may be appreciated by noting how frequently the formulation of problems in applied mathematics involves derivatives. It is then natural to anticipate that the solutions of such problems will involve integrals. Students are expected to know that for most integrals no representation in terms of elementary functions is possible, and approximate integration becomes necessary. For example, the integrals $\int \frac{\sin x}{x}\,\mathrm{d}x$ and $\int e^{-x^2}\,\mathrm{d}x$ are difficult to find analytically. Students should be taught that polynomial approximation like the Lagrange Interpolating Polynomial method serves as the basis for the two integration formulae, namely Trapezoidal rule and Simpson's rule, studied in this course, the main idea being that if $p(x)$ is an approximation to $f(x)$, then $$\int_a^b p(x)\,\mathrm{d}x \approx \int_a^b f(x)\,\mathrm{d}x$$
16.2 Trapezoidal Rule (a) Derivation of the trapezoidal rule	6	One way of motivating students to the learning of Trapezoidal rule is by appealing to the geometry of the rule, which uses a series of trapezoids to approximate the area in question. A known definite integral such as $\int_1^2 x^2\,\mathrm{d}x$ can be used to demonstrate that the rule works well providing good accuracy if the number of trapezoids is sufficient. It is easy for students to derive the Trapezoidal rule for the integral $\int_a^b f(x)\,\mathrm{d}x$ with n trapezoids to be

Detailed Content	Time Ratio	Notes on Teaching
		$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{w}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots\ldots+2f(x_{n-1})+f(x_n)\right]$$ $$=\frac{w}{2}\left[f(x_0)+2\sum_{k=1}^{n-1}f(x_k)+f(x_n)\right]$$ where $x_0=a$, $x_n=b$ and $w=\frac{b-a}{n}$. Alternatively, the derivation may be done by linear interpolation in each interval. For example, in the subinterval $[x_0, x_1]$. $$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x \approx \int_{x_0}^{x_1} p(x)\,\mathrm{d}x$$ $$=\int_{x_0}^{x_1}\left[f(x_0)\left(\frac{x-x_1}{x_0-x_1}\right)+f(x_1)\left(\frac{x-x_0}{x_1-x_0}\right)\right]\mathrm{d}x$$ $$=\frac{w}{2}\left[f(x_0)+f(x_1)\right] \text{ where } w=x_1-x_0$$ By summing the area in all the subintervals, students could obtain the same formula for Trapezoidal rule as before.
(b) Estimation of the error		It is interesting to note what kind of accuracy may be expected for a given function. Teachers may guide students to derive the maximum error as follows. Consider the ith trapezoid of a trapezoidal integration, which lies between x_{i-1} and x_i, two points at a distance $w=\frac{b-a}{n}$ apart. Assume $F(x)$ is the primitive function of $f(x)$. Then, teachers may ask students to give the exact value of the integral $\int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x$ $(=F(x_i)-F(x_{i-1}))$ and the calculated value $\left(=\frac{w}{2}\left[f(x_{i-1})+f(x_i)\right]\right)$. Defining the error on this trapezoid as $$E_i=\frac{w}{2}\left[f(x_{i-1})+f(x_i)\right]-\left[F(x_i)-F(x_{i-1})\right]$$

Detailed Content	Time Ratio	Notes on Teaching

and using Taylor's series expansion of $f(x_{i-1})$ and $F(x_{i-1})$ about x_i students could be guided to discover

$$E_i \approx \frac{w^3}{2} f''(\varepsilon_i)$$

By summing the errors in all the subintervals, students should be able to obtain the total error as

Maximum total error = $|E_T| = (b-a)\frac{w^2}{12}M$ where $M = \max|f''(\varepsilon)|$ for ε in the range of integration.

(c) Application of trapezoidal rule

Applications should be stressed. Common examples including computation of area, work done by variable force and distance covered by a particle with given velocity, can be given to students for illustration. The following are some of them.

Example 1

A curve is given by the points tabulated in the table. (t is the time travelled and v is the speed.) Students may be required to calculate the distance covered between t = 0 and t = 4.0.

t (hours)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
v (kmh^{-1})	23	20	15	11	12.5	15	18	20	22

Example 2

Use the trapezoidal rule with four equally spaced ordinates to estimate the value of $\int_0^1 e^{\sqrt{x}}\,dx$ to 3 significant figures.

Example 3

How small an interval w would be required to obtain In 2 correct to 4 decimal places?

Example 4

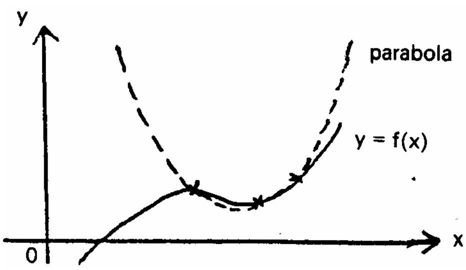
Evaluate the integral $\int_1^2 \frac{1}{1+x^2}\,dx$ using Trapezoidal rule with an accuracy of 0.001 and check the answer against the true value.

Detailed Content	Time Ratio	Notes on Teaching

16.3 Simpson's Rule — Time Ratio: 6

(a) Derivation of Simpson's rule

The derivation can be done using the second-degree L.I.P., but the geometrical meaning must be emphasized. Students should be able to realize that approximation by a series of parabolic segments would, in general, more closely match a given curve $y = f(x)$ than would the straight lines in the Trapezoidal method.



Students may be guided to derive Simpson's rule with 2n strips as

$$\int_a^b f(x)\,dx \approx \frac{w}{3}\left[f(x_0) + 4\sum_{k=1}^{n} f(x_{2k-1}) + 2\sum_{k=1}^{n-1} f(x_{2k}) + f(x_{2n})\right]$$

where $x_0 = a$, $x_n = b$ and $w = \frac{b-a}{n}$.

Teachers should also remind students that the number of subintervals (or strips) used in Simpson's rule must be even, but there is no such restriction in Trapezoidal rule.

(b) Estimation of the error

In a similar way to Trapezoidal rule, the truncation error of Simpson's rule may also be derived using Taylor's series expansion of $f(x)$ and the usual assumptions being made as before. It is not too difficult though a bit more tedious for students themselves, with some guidance from teachers, to arrive at the maximum total error term

$|E_T| = (b-a)\frac{w^4}{180}M$ where $w = \frac{b-a}{2n}$ and $M = \max|f^{(4)}(\varepsilon)|$ for ε in the range of integration.

Detailed Content	Time Ratio	Notes on Teaching
(c) Application of Simpson’s rule		Basically, the kinds of problems resemble those of Trapezoidal rule. Here the emphasis, apart from the application of the technique itself, should be placed on the comparison of the degree of accuracy between the two formulae. Two examples follow. *Example* 1 Evaluate the integral $\int_0^{\frac{\pi}{6}} \ln(\cos x)\,dx$ by Simpson’s rule. Find the least number of strips required so that the error is less than 10^{-6} in magnitude. *Example* 2 Find the value of the integral $\int_1^2 \frac{1}{1+x^2}\,dx$ using both Trapezoidal and Simpson’s rule with 6 strips. Compare their accuracy with the standard result.
	13	