UNIT 17: Numerical Solution of Equations

Specific Objectives:

1. To learn fixed point iteration method, Newton's method, Secant method and method of false position.

2. To acquire the skill in using the relevant methods to find approximate roots of equations, and compute the errors of the roots.

Detailed Content	Time Ratio	Notes on Teaching
		In scientific and engineering work, a frequently occurring problem is to find the roots of equations of the form $f(x)=0$. For quadratic, cubic or quartic equations, exact solutions are readily obtained using algebraic methods. However, when $f(x)$ is a polynomial of higher degree than four or a transcendental function such as $e^x - 4\cos x$, algebraic methods are not easily available. It is natural to find the solution by approximate methods.
17.1 Method of Fixed-point Iteration (a) Algorithm of the method	7	Students should be taught that the method consists of several steps. 1. Rearranging the equation $f(x)=0$ in the form $x=g(x)$, where $g(x)$ is called the iteration function. 2. Making an initial guess x_0 by basing on a sketch of the appropriate graph(s) where necessary. 3. Obtaining a sequence of $x_0, x_1, x_2, \ldots\ldots, x_n, \ldots\ldots, \alpha$ by substituting in the equation as follows. $x_1 = g(x_0)$ $x_2 = g(x_1)$ $x_3 = g(x_2)$ $\vdots$ $x_i = g(x_{i-1})$ $x_{i+1} = g(x_i)$ $\vdots$ It is hoped that a fixed point *a* is obtained such that $\alpha = g(\alpha)$. 4. An approximate root of $f(x)=0$ is the fixed point α. Teachers may demonstrate the method using some examples of equations with suitable initial guesses and iteration functions that converge.

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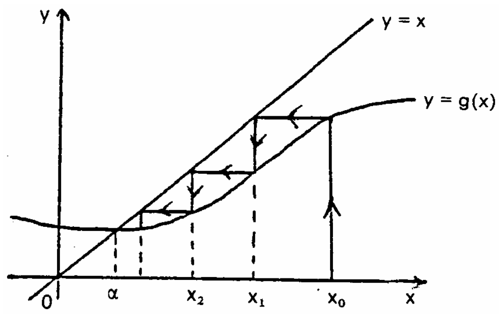
		The geometry of the method should be discussed. 
(b) The condition of convergence		Using an example like $x^2 - 5x + 4 = 0$ and an iteration function $g(x) = \frac{x^2+4}{5}$, teachers may ask students to find the roots using initial guesses $x_0 = 2$ and 5 respectively. Students would find that for $x_0 = 2$, they get the root near 1 but for $x_0 = 5$, the procedure diverges. The discussion of convergence then becomes natural. It is worthwhile to geometrically discuss monotonic and oscillating convergence and divergence. Mathematical treatment of monotonic and oscillating convergence as well as divergence should be briefly mentioned. To discuss convergence mathematically, the use of mean value theorem is essential. It is advantageous for students to know that for the fixed-point iteration algorithm to be useful the following are usually needed. 1. There is an interval *I* = [*a*, *b*] such that for all *x* in *I*, $g(x)$ is defined and $g(x)$ in *I*. 2. The iteration function $g(x)$ is continuous on *I*. 3. The iteration function is differentiable on *I* and for all *x* in *I*, there exists a real number *K* such that $\lvert g'(x)\rvert \le K < 1$. Exercises on proofs like the following can be given.

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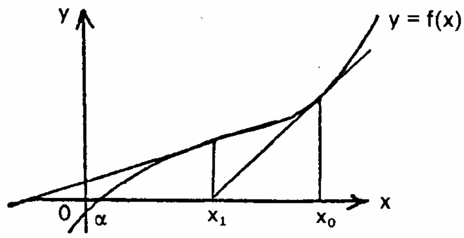
		Example It is given that $x = g(x)$ has exactly one root α in [*a*, *b*], and that $\lvert g'(x)\rvert \le K$ for any *x* in [*a*, *b*]. Suppose further that $x_{n+1} = g(x_n)$ where $a \le x_n \le b$, *n* = 1, 2, 3, ... Students may be required to show that $\lvert x_{n+1} - \alpha\rvert \le K^{n+1}\lvert x_0 - \alpha\rvert$ and deduce that, if *K* < 1, then the sequence $\{x_n\}$ converges to α.
(c) Estimation of error		The error of the n^{th} approximation x_n , ε_n , where $\varepsilon_n = \lvert x_n - \alpha\rvert \le \frac{K^n}{1-K}\lvert x_1 - x_0\rvert$ can be derived and discussed. It may be seen that the smaller the value of K, the faster the rate of convergence. For abler students, teachers may discuss the order of convergence using Taylor's series expansion of the error about the fixed point α $\varepsilon_{n+1} = g'(\alpha)\varepsilon_n + g''(\alpha)\frac{\varepsilon_n^2}{2!} + g'''(\alpha)\frac{\varepsilon_n^3}{3!} + \cdots$ Rigorous treatment, nonetheless, should not be attempted.
17.2 Newton's Method (a) Algorithm of the method	5	The algorithm may be derived geometrically.  Alternatively, it may also be derived using Taylor's series expansion of $f(x_{n+1})$ about x_0 as $f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + f''(x_n)\frac{(x_{n+1} - x_n)^2}{2!} + \ldots\ldots$

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		When $x_{n+1} \to \alpha$ and $x_{n+1} - x_n \to 0$, students should be able to arrive at $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ It is necessary that students know that Newton's method is a special case of the fixed-point iteration.
(b) The condition of convergence and error estimation		As such, the condition of convergence is the same as the general fixed-point iteration method. It is interesting to discuss the rate of convergence of Newton's method as compared with that of the general method. Students could easily find that $g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$ and $g'(\alpha) = 0$ if α is a simple root, and that $\varepsilon_{n+1} \approx g''(\alpha)\frac{\varepsilon_n^2}{2!}$. Examples in which Newton's method excels others should be given. An example follows. *Example* The root in [0, 0.8] of the equation $x^3 + 2x - 1 = 0$ is to be determined by an iteration formula $x_{n+1} = \frac{1}{2}\left(1 - x_n^3\right)$. Students may be asked to find α with $x_0 = 0$ and then required to do the same using Newton's method. Finally they may be required to account for the faster rate of convergence of the latter over the former, It is profitable for students to know that when α is a double root Newton's method is not that fast and it is preferable to discuss the pitfalls for the method as well in order to make the study of Newton's method more complete.
(c) Application of Newton's method		Problems on polynomial equations of degree higher than two and transcendental equations are relevant. For example, finding the root between *x* = 0 and *x* = 1 of the equation $2x^3 + x^2 - 20x + 20 = 0$ with an accuracy of 10^{-6}, and deriving Newton's formula $x_{n+1} = x_n - \frac{x^k - a}{kx_n^{k-1}}$ for finding the *k*th root of *a* are common questions.

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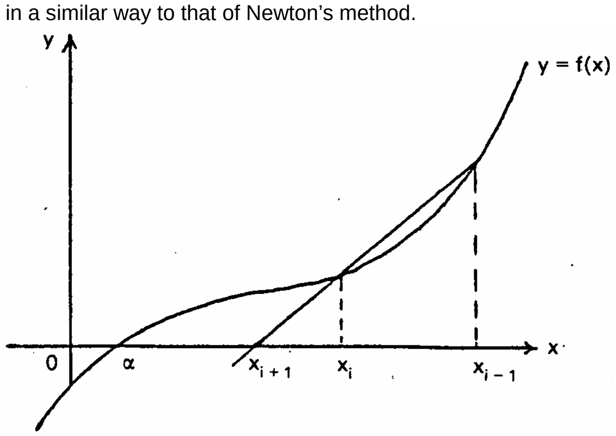
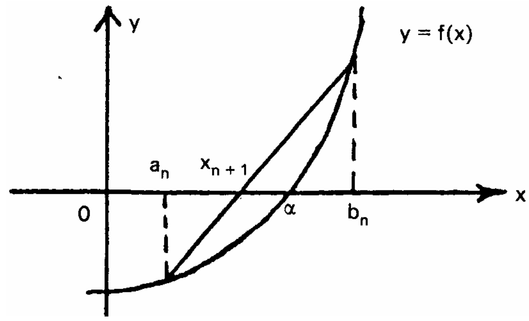
17.3 Secant Method		The secant method is another method for finding the roots of $f(x)=0$. It often converges almost as fast as Newton's method, but avoids the need for calculating the derivative $f'(x)$. Instead of using a tangent line, a secant line is used.
(a) Derivation of the secant method		The derivation of the algorithm can be easily done by appealing to the geometry of the method in a similar way to that of Newton's method.  The analogous formula is $x_{n+1}=x_n-f(x_n)\dfrac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}$ with the slope of the secant $\dfrac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$ replacing the slope of the tangent, $f'(x)$. Students should be able to see that the secant method requires two initial guesses at which the functional values need not be of different signs. In-depth treatment of rate of convergence and error estimation is not needed, When such is required for calculation, the relevant formulae will be given in the questions.

Detailed Content	Time Ratio	Notes on Teaching
(b) Application of the secant method		Exercises are similar to those for Newton's method.
17.4 Method of False Position	2	This is still another method for finding a root of the equation $f(x)=0$ lying in the interval [a, b]. The method is similar to the Bisection method (which students have learned in S.5) in that intervals [a_n, b_n] are generated to bracket the root, and the method is also similar to the Secant method in the manner of obtaining new approximate iterates. The method is also named Regula Falsi.
(a) Derivation of the method of false position		Assuming that the interval [a_n, b_n] contains a root of f(x) = 0 and with the help of a diagram like the following  teachers can guide students to compute the value of the x-intercept of the line joining the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$. This point labelled x_{n+1} will be found to be $x_{n+1}=a_n-\dfrac{f(a_n)(b_n-a_n)}{f(b_n)-f(a_n)}$ or $x_{n+1}=b_n-\dfrac{f(b_n)(b_n-a_n)}{f(b_n)-f(a_n)}$. Students should find it easy to complete the algorithm by defining $a_{n+1}=a_n$ and $b_{n+1}=x_{n+1}$ if $f(x_{n+1})f(a_n)<0$ and defining $a_{n+1}=x_{n+1}$ and $b_{n+1}=b_n$ if $f(x_{n+1})f(a_n)>0$.

Detailed Content	Time Ratio	Notes on Teaching
(b) Application of the method of false position		Students should be reminded that two initial guesses are required by the method and that the functional values at these initial guesses must be of different signs for the method to be applicable. Exercises similar to those for Newton's method and Secant method are appropriate. Following are some examples. *Example* 1 Show that the equation $x^3 + 3x - 12 = 0$ has exactly one root in the interval [1, 2] and find this root by the method of false position correct to 3 decimal places. *Example* 2 The method of false position is used to find the root of $x^3 = 2x + 5$ in the interval [2, 3]. Show that the sequence of iterates $\{x_i\}$ is given by $x_{i+1} = 3 - \frac{16x_i - 48}{x_i^3 - 2x_i - 21}$ i = 1, 2, 3, …… Hence find the root correct to 4 significant figures. As in the case of Secant method, detailed treatment of convergence and error estimation is not recommended.
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