**UNIT 18: Introductory Probability Theory**

*Specific Objectives:*

1. To know the use and the importance .of probability in daily life.
2. To learn the basic laws of probability and their applications in real-life problems.



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| **18.1 Basic Definitions**<br>Sample points, sample space, equiprobable space, probability of events | 3 | Students may have learnt some of the concepts in this sub-topic. However, since these concepts are essential for the study of this topic area, a brief discussion is worthwhile to recall and consolidate what they have learnt.<br><br>Students should be familiar with the meaning of sample space and event. Sample points may be new to some students. Examples such as throwing of coins should be discussed and teachers are advised to emphasize to students that sample space in an experiment is not unique. For example, the two possible sample spaces in a throw of 3 coins are {(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H). (T, H, T), (T, T, H), (T, T, T)} and {no head, 1 head, 2 heads, 3 heads}<br><br>Teachers should remind students of the meaning of equiprobable space. In the previous example, the former sample space is equiprobable while the latter is not. The following example may also be used for illustration.<br><br>*Example*<br>A man who goes to work every day mayor may not be killed in a traffic accident. However, the probability that he will be killed in a traffic accident when he is going to work one day is not equal to } because the two outcomes ('killed' and 'not killed') are not equally likely to happen.<br>Once the concept is clarified, students can be led to recall the definition of probability:<br>$P(E)=\frac{n(E)}{n(S)}$ where E is the event, S is the equiprobable sample space<br>Teachers should emphasize that $0 \le P(E) \le 1$. The cases of certainty ($P(E)=1$) and impossibility ($P(E)=0$) should also be discussed. |



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| **18.2 Ways of Counting** | 5 | To facilitate the finding of the number of outcomes, different ways of counting should be introduced. One of the ways of counting is the Multiplication Principle as shown below.<br><br>$n_1$ elements $a_1, a_2, \ldots\ldots, a_{n_1}$<br>$n_2$ elements $b_1, b_2, \ldots\ldots, b_{n_2}$<br>$\vdots$ $\vdots$<br>$n_r$ elements $x_1, x_2, \ldots\ldots, x_{n_r}$<br><br>It is possible to form $n_1, n_2, \ldots\ldots, n_r$ ordered $r$ tuples ($a_{j_1}, \ldots\ldots, x_{j_r}$) containing one element of each kind.<br><br>*Example*<br>Suppose that the students of a certain school are classified according to sex, age and the house to which they belong. If there are 4 houses, 5 age groups, then there are 2 × 4 × 5 = 40 groups in all.<br><br>Other ways of counting include Permutation and Combination and their related formulae such as<br>$P_r^n=\frac{n!}{(n-r)!}$, $C_r^n=\frac{n!}{(n-r)!r!}$<br>$C_r^n=C_{n-r}^n$ and $C_r^n+C_{r+1}^n=C_{r+1}^{n+1}$<br>should be clearly taught. Emphasis should be laid on the difference between Permutation and Combination. Examples such as finding the probability that each of n cells will be occupied when n balls are randomly placed in them, and the probabilities of winning different prizes in Mark Six will help students to consolidate the idea of the ways of counting.<br><br>Students are only required to master the basic techniques of using permutations and combinations to solve simple problems. It is not worthwhile for the teachers to put too much effort on this topic. |

| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | For the abler students, teachers may discuss with them the following general formula.<br>Among $n$ elements, of which $p_1$ elements (type 1) are similar, $p_2$ elements (type 2) are similar, …… $p_k$ elements are similar (type $k$), they can be arranged (orderly) in $\frac{n!}{p_1!p_2!\ldots p_k!}$ ways.<br><br>Other applications, such as 'simple hypergeometric probabilities' may also be discussed, but knowledge of the term is not necessary. Daily life examples such as Quality Inspection problem and Estimation of number of fish in a pond could be discussed. The following are two examples.<br><br>*Example 1*<br>In a shipment of n video tape recorders containing r defective items, a sample of $p$ items are chosen at random without replacement for inspection ($r < n$ and $p < n$). Students are required to calculate the probability that there are exactly $q$ defective recorders if $q < r$ and $q < p$.<br><br>Students may be asked to calculate the probability that at least 2 defective video tape recorders are found if $n = 80$, $r = 10$, $p = 15$.<br><br>*Example 2*<br>From an estimated population of $N$ fish in a pond, a sample of $r$ is caught, marked and put back to the pond. After the population is thoroughly mixed, a second sample of $r$ is taken again.<br><br>It is not difficult to find the probability that there will be n marked fish in the second sample. Also teachers can discuss with students how to find the most probable number of fish in the pond. |
| **18.3** **Probability Laws**<br>Sum and product rules<br>Mutually exclusive events<br>Independent events<br>Conditional probability | 5 | Students should have no difficultly to learn the definitions of mutually exclusive events and independent events. The corresponding sum and product rules i.e. $P(E\cup F)=P(E)+P(F)$ and $P(E\cap F)=P(E)\cdot P(F)$ should be discussed. Typical examples such as tossing coins, drawing balls and throwing dice can help students to recall the above concepts. |



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | Teachers should teach the meaning and notation of conditional probability through examples.<br><br>Now, teachers should introduce to students the cases of dependent events and events that are not mutually exclusive. The sum and product rules then respectively become $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ and $P(A\cap B)=P(A)\cdot P(B\,\|\,A)$.<br><br>The general formula for $P(E_1\cap E_2\cap\ldots\ldots\cap E_n)$ may be discussed. Also it is worthwhile to mention that two events A, B are independent if and only if $P(A\,\|\,B)=P(A)$ or $P(A\cap B)=P(A)\cdot P(B)$.<br><br>In handling problems of which the outcomes can occur in a finite number of ways, tree diagram is an efficient way to enumerate all the possible outcomes. Appropriate examples should be chosen to demonstrate the use of the rules. The following are some possible examples:<br><br>*Example 1*<br>A bag A contains 4 red and 6 black balls, and a bag B contains 6 red and 4 black balls. A ball (first ball) is drawn at random from A and placed In B. After mixing, a ball (second ball) is drawn at random from B and placed in A. Finally, a ball (third ball) is drawn from A.<br>In this example, teachers may ask students to find the probability that ith ball is red (i = 1,2,3). Students may be reminded that all the possible outcomes can be represented graphically using a tree diagram.<br><br>*Example 2*<br>John is available to meet his friend at home during the weekend (Friday to Sunday). Given that the probability Mary goes to visit John on Friday is $\frac{1}{q}$. On each of the other two days, the probability that she goes, given that she has gone the previous day, is $\frac{1}{m}$ and the probability that she goes, given that she has not gone the previous day, is $\frac{1}{n}$.<br>The possible questions in this example include finding the probability that Mary goes on Sunday, and finding $P(B\,\|\,A)$ and $P(B\,\|\,\bar{A})$ where $A$ denotes the event that Mary goes on Friday while $B$ the event that she goes on Sunday. Teachers can lead students to draw the tree diagram which will definitely help students to exhaust all the possible cases and to solve the problem. |

| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| **18.4 Bayes' Theorem** | 4 | After the students are acquainted with the conditional probability, teachers can go further to Bayes' Theorem which is stated as follows:<br>Let the sample space $\Omega$ be partitioned by mutually exclusive events $A_1, A_2, \ldots\ldots, A_n$<br>Let B be another event so that $P(B) \neq 0$.<br>$P(A_j \mid B) = \dfrac{P(A_j \cap B)}{P(B)} = \dfrac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)}$<br>As students are not expected to derive the theorem, teachers may use numerical examples to explain the meaning of each term in this theorem. In most of the problems involving Bayes' theorem, tree diagram is a commonly used technique to help students to solve the problems. The following is one of the possible examples.<br>*Example*<br>Three urns contain respectively 6 black and 9 white balls, 12 black and 3 white balls, 8 black and 7 white balls. One urn is chosen at random and a ball is drawn from it. Teachers may ask students to draw the tree diagram and discuss with them how P(black) can be obtained. Then, students should have no difficulty to get the value of<br>P(ball came from second urn \| black). |
| **18.5 Recurrence Relation** | | Teachers should help students to identify the sort of situation where the probability of the nth event depends on the result of the previous event(s). Students are expected to form an equation relating them. The following examples may be useful in explaining the concept of recurrent situations.<br>*Example 1*<br>There are two new brands of soft drink to be introduced on the market. In consideration of the different ways of packing, there is a probability of 0.55 that a girl will choose Brand X and a probability of 0.45 that she will choose Brand Y. It is assumed that she drinks one and only one of the two brands of soft drink each day. If the last brand she chose was X, there is a probability of 0.6 of choosing Brand X on the subsequent day and a probability of 0.4 of choosing Brand Y. On the other hand, if she last drank Y, she will choose X with a probability of 0.3 and Y with a probability of 0.7. |



| Detailed Content | Time Ratio | Notes on Teaching |
|---|---|---|
| | | In this example, a tree diagram can be used to show the relationship between $P_{n-1}$, $1-P_{n-1}$, $P_n$ and $1-P_n$ where $P_n$ denotes the probability that she buys X on the nth occasion. From the tree diagram, it is easy to express $P_n$ in terms of $P_{n-1}$. Most of the students should be able to express $P_n$ in terms of $P_1$ and *n*. Students may be interested to know that the ultimate share of market of brand *X* is actually $\lim_{n\to\infty} P_n$.<br>*Example 2*<br>From a bag containing 4 white balls and 9 black balls, one ball is drawn and replaced, this operation being performed n times. On is the probability that no two consecutive draws produce two black balls. Teachers can show to the students the way to obtain a relation between $Q_n$, $Q_{n-1}$ and $Q_{n-2}$ ($n \geq 3$). They may also ask students to write down $Q_1$, $Q_2$ and $Q_3$ and hence calculate $Q_4$, $Q_5$ .......<br>Markov Chain and difference equations are not expected. |
| | 19 | |