UNIT 21: Statistical Inference

Specific Objectives:

1. To estimate a population mean from a random sample.
2. To recognize the confidence interval for the mean of a normal population with known variance.
3. To recognize hypothesis testing and Type I and Type II errors.

Detailed Content	Time Ratio	Notes on Teaching
21.1 Basic Concept	4	Teachers should introduce the terms ‘population’, ‘sample’, ‘random sample’, ‘population parameter’ and ‘sample statistic’. *Example 1* The height of a particular species of plants follows a normal distribution with mean 20 cm and standard deviation 8 cm. Students are expected to see that the sample mean of a random sample of 10 plants is normally distributed with mean 20 cm and standard deviation $\frac{8}{\sqrt{10}}$ cm *Example 2* It is known that 3% of electric bulbs will be broken on delivery. If 1 000 electric bulbs were sent out, find the probability that 5% or more will be broken. In this example, teachers may guide students to consider the number of broken bulbs X which is a binomial random variable, i.e. X ~ B (1 000, 0.03) It can be approximated by the normal distribution, i.e. x ~ N ((0.03)(1 000), (1 000)(0.03)(0.97)) Students can easily see that the required probability is $P(X > 49.5)$

Detailed Content	Time Ratio	Notes on Teaching
		Alternatively, teachers may make use of the concept of sample proportion $P_s = \frac{X}{n}$ which can be approximated by $P_s \sim N\left(P, \frac{pq}{n}\right)$ where $q = 1 - p$ In this case, the required probability is $P\left(P_s \geq 0.0495\right)$
21.2 Estimate of a Population Mean from a Random Sample	5	It is worthwhile, at this stage, for teachers to introduce the concept of estimation of an unknown population parameter from a sample statistic. Examples like estimating a population mean μ by using a sample mean $\bar{x}$ can be used for illustration. Teachers should indicate to students that there may be several sample statistics which can be used as estimators. For examples, sample mean, median and mode could also be used to estimate the population mean μ. In view of this, students are expected to know that the best estimator b among the various sample statistics used to estimate the population parameter β should (1) be unbiased, i.e. $E(b) = \beta$ and (2) has the smallest variance. *Example* 1 Let X_1, X_2, X_3 be random samples taken from a population with mean μ and variance σ^2. $T_1 = \frac{X_1 + X_2 + X_3}{3}$ $T_2 = \frac{X_1 + 2X_2}{3}$ $T_3 = \frac{X_1 + 2X_2 + 3X_3}{3}$ are estimators of μ.

Detailed Content	Time Ratio	Notes on Teaching
		Students can easily reach the results that T_1 and T_2 are unbiased estimates of μ, and T_1 is the best estimator among T_1, T_2 and T_3, *Example 2* Two random sample of sizes n and $3n$ are taken from normal populations with mean μ and 3μ and variances σ^2 and $3\sigma^2$ respectively. If $\bar{X}_1$ and $\bar{X}_2$ are the sample means, show that the estimator $a\bar{X}_1 + b\bar{X}_2$ is an unbiased estimator for μ if $a + 3b = 1$. Students are expected to know that the sample mean $\bar{x}$ is the most efficient estimator for the population mean, but a proof is not necessary. *Example 3* The following shows a random sample of size 7. 9.30, 9.61, 8.27, 8.90, 9.14, 9.90, 9.10 The most efficient estimate of the population mean is then given by $\frac{\sum x}{n} = \frac{9.30 + 9.61 + \ldots + 9.10}{7} = 9.17$ approximately *Example 4* Let p be the probability of obtaining a '6' when a loaded die is rolled. John carried out an experiment to find p by rolling the die 100 times and recorded 20 '6's. Mary repeated the same experiment 200 times independently and recorded 50 '6's. Students are expected to evaluate John's and Mary's estimate for p respectively by using the formula $E\left(P_s\right) = p$ For abler students, teacher may guide them to improve the estimation by pooling the two estimates. (The pooled estimator for p is given by the formula $\hat{p} = \frac{n_1 P_{s_1} + n_2 P_{s_2}}{n_1 + n_2}$)

Detailed Content	Time Ratio	Notes on Teaching
21.3 Confidence Interval for the Mean of a Normal Population with Known Variance	6	In general, teachers should point out that an interval estimate of an unknown population parameter (e.g. the mean μ) is a random interval constructed so that it has a given probability of including the parameter. Students should also be told that the most commonly used confidence intervals are the 95% confidence interval and the 99% confidence interval. Teachers should indicate to students how the confidence interval can be used to estimate the mean of a normal population whose variance σ^2 is known. Students should realize that if the sample size is n, the sample mean $\bar{X}$ is normally distributed with mean μ and variance $\frac{\sigma^2}{n}$. Standardizing, $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ where $Z \sim N(0, 1)$. Students should know that the central 95% of $N(0, 1)$ lies between ± 1.96. Thus, $P(-1.96 \le \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \le 1.96) = 0.95$ or $P(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}) = 0.95$ Thus, if $\bar{x}$ is a value of $\bar{X}$, then $\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$ is a 95% confidence interval of μ. *Example 1* The masses of a random sample of 12 articles in grams are 200, 204, 196, 198, 210, 189, 197, 221, 205, 203, 196, 199. If this sample came from a normal population with standard deviation 10 g, students should have no difficulty to obtain a 95% confidence interval for the mean mass of the population.

Detailed Content	Time Ratio	Notes on Teaching
		Example 2
21.4 Hypothesis Testing	6	

Example 2

A machine produces 10 000 items, 300 of which are defective. To find a 95% confidence interval for the probability p that an item is defective, teachers should guide students to take $\frac{300}{10\ 000}$ as an estimate of P and $\frac{\left(\frac{300}{10\ 000}\right)\left(\frac{9\ 700}{10\ 000}\right)}{10\ 000}$ as an estimate of $\frac{pq}{n}$.

By taking the approximated distribution $P_s \sim N\left(P, \frac{pq}{n}\right)$, students can solve the problem in a similar way.

21.4 Hypothesis Testing — Time Ratio: 6

Teachers may introduce the concept of hypothesis testing by using real-life examples such as deciding on the basis of sample data whether the true average lifetime of a certain kind of electric light bulb is at least 2 000 hours etc. The terms 'null hypothesis', 'alternative hypothesis' and 'level of significance' together with their corresponding notations (i.e. H_0, H_1 and α) should be clearly explained with examples provided for illustration. Students are also expected to recognize the terms 'critical point', 'critical region', 'region of acceptance' and 'region of rejection'.

The following show some typical examples.

Example 1

The lengths of nails produced by a particular machine are normally distributed with variance 0.26 mm^2. The machine had been set to produce nails with a mean length of 30 mm, but now there is some doubt about the nail length produced recently. A random sample of 50 nails was found to have a mean length of 30.2 mm.

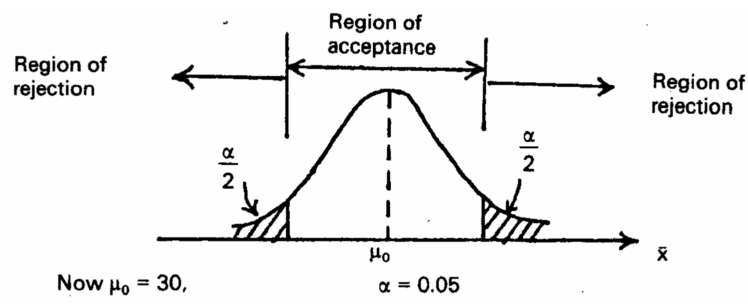
(a) Test at the 5% level of significance whether the mean length of 30 mm is accepted or not.

(b) Test at the 5% level of significance whether the mean length of nails produced by the machine is greater than 30 mm.

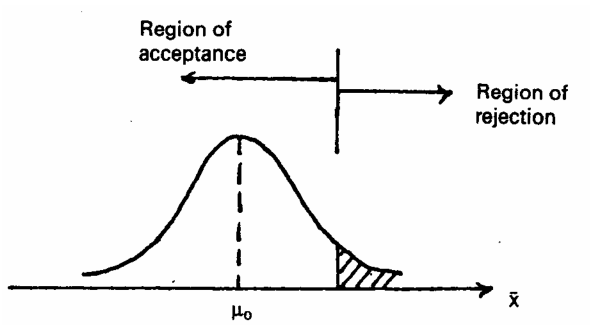
Detailed Content	Time Ratio	Notes on Teaching

Teachers can discuss with students how to set the alternative hypothesis in *(a)* and *(b).* This leads to the concept of two-tailed and one-tailed test:

(a) Two-tailed test



(b) One-tailed test



Detailed Content	Time Ratio	Notes on Teaching

Example 2

The producer of a certain brand of canned dog food claims that 75% of puppies prefer his brand. A random sample of 200 dogs was tested. 135 chose his brand. Test the producer's claim at the 5% level.

In this example, it is important for teachers to point out that the percentage of puppies can be regarded as the proportion of success, P_s, which has a distribution

$$P_s \sim N\left(P,\ \frac{pq}{n}\right) \text{ approximately.}$$

21.5 Type I and Type II Errors | Time Ratio: 6

When students are familiar with the concept of and procedures of hypothesis testing, teachers may guide them to summarize the four possible conclusions of a test. They are tabulated as follows:

Real Situations	Test Results	Remarks
1. H_0 is true	Accept H_0	Correct decision
2. H_0 is true	Reject H_0	A Type I error has been committed.
3. H_0 is false	Accept H_0	A Type II error has been committed.
4. H_0 is false	Reject H_0	Correct decision

It is obvious for students to notice that

P(Type I error) =. P(rejecting H_0 | H_0 is true)

and P(Type II error) = P(accepting H_0 I H_0 is false)

The following are some typical examples.

Example 1

A box is known to contain either (H_0) 10 white balls and 90 black balls or (H_1) 50 white balls and 50 black balls. In order to test hypothesis H_0 against hypothesis H_1, four counters are drawn at random from the box without replacement. If all four counters are black, H_0 is accepted. Otherwise, it will be rejected.

Detailed Content	Time Ratio	Notes on Teaching

In this example, students should be able to find the probability of the Type I and Type 11 errors for this test by using the above relations.

Example 2

The ingredients for concrete are mixed together to obtain a mean breaking strength of 2 000 newtons. If the mean breaking strength drops below 1 800 newtons, then the composition must be changed. The distribution of the breaking strength is normal with standard deviation 200 newtons. Samples are taken in order to investigate the hypothesis:

$$H_0: \mu = 2000 \text{ newtons}$$
$$H_1: \mu = 1\,800 \text{ newtons}$$

How many samples must be tested so that

P(Type I error) = 0.05

and P(Type II error) = 0.1?

Example 3

The calibration of a scale is to be checked by weighing a 10 kg test specimen 25 times. Suppose the results 6f different weightings are independent of one another and that the weight on each trial is normally distributed with $\sigma = 0.200$ kg. Let μ denote the true weight reading on the scale.

(a) What hypothesis should be tested?

(b) Suppose the scale is to be recalibrated if either $\bar{x} \geq 10.1032$ or $\bar{x} \leq 9.8968$. What is the probability that recalibration is carried out when it is actually unnecessary? Which type of error would that be?

(c) What is the probability that recalibration is judged unnecessary when in fact μ = 10.1? When μ = 9.8? Which type of error are these?

Time Ratio total: 27