UNIT 4: Trigonometry

Specific Objectives:

1. To understand the six trigonometric functions of the general angle and their graphs.
2. To understand and apply the compound angle formulae and sum and product formulae.
3. To find the general solution of trigonometric equations.
4. To acquire skills in solving harder problems in two and three dimensions.

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Detailed Content	Time Ratio	Notes on Teaching
4.1 Radian Measure	~~2*~~ 3*	Students should understand the meaning of a radian. They should be able to derive the formulae s = rθ and $A = \frac{1}{2}r^2\theta$ and to find the arc length and the area of a sector respectively. Students should be able to convert any angles measured in degrees to radians and vice versa. Adequate practice on evaluation of trigonometric functions and formulae involving angles in radian measures should be provided.
4.2 The Six Trigonometric Functions of Angles of Any Magnitude and their Graphs	~~4*+4~~ 5*+5	Students are expected to be familiar with the sine, cosine, tangent functions and their graphs in the interval from 0 to 2π. The domain of these functions can be extended to the whole set of real numbers. Students should discover that sine, cosine and tangent are periodic functions with period 2π, 2π, and π respectively. Teachers may define the functions cosecant, secant and cotangent by means of the unit circle. Students should note that $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

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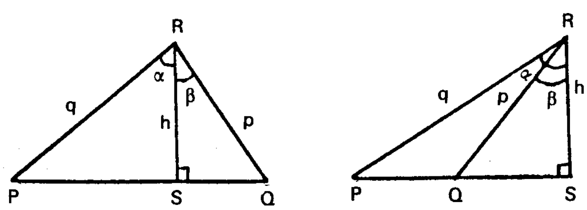
Detailed Content	Time Ratio	Notes on Teaching
		$\cot\theta = \frac{1}{\tan\theta}$ $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ Teachers should show how to simplify the six trigonometric functions at $\frac{n\pi}{2} \pm \theta$ for odd and even n. Graded exercises involving these relations and identities should be given. Students are also expected to discover that the functions cosecant, secant and cotangent are also periodic functions and have periods of 2π, 2π and π respectively. *Example 1* Prove the identity $\sec^2\theta\operatorname{cosec}^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta$ *Example 2* Given that $\frac{\sin^2\theta}{1 + 2\cos^2\theta} = \frac{3}{19}$, where $\frac{\pi}{2} < \theta < \pi$, find the value of $\frac{\sin\theta}{1 + 2\cos\theta}$.
4.3 Compound Angles	~~10~~ 9	Teachers may introduce the formulae $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ by considering the area of triangle PQR in each of the following diagrams:



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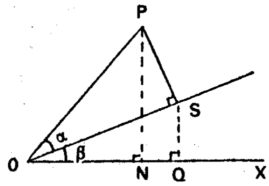
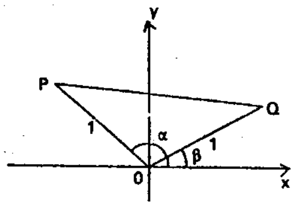
Detailed Content	Time Ratio	Notes on Teaching

Alternatively, teachers may show that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ by considering the projection of a line segment OP on the line OX



or $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ by applying cosine formula (if students have learnt it) to the triangle OPQ in the figure below.



Students should note that the formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$ are true for angles A and B of any magnitude. Ample examples and exercises should be provided to help students get familiar with these formulae. Students may also be encouraged to derive for themselves the formulae.

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

and the sum and product formulae:

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

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Detailed Content	Time Ratio	Notes on Teaching

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

other formulae such as the following can also be derived.

Double angle formulae

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

Triple angle formulae

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

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Detailed Content	Time Ratio	Notes on Teaching

Half angle formulae

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

Teacher should emphasize on the applications of these formulae

Examples such as the following can be introduced:

Example 1

Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, show that $\tan\frac{\pi}{8}$ is a root of the equation $x^2+2x-1=0$ and hence find the value of $\tan\frac{\pi}{8}$ in surd form.

Example 2

Solve the equation $\cos 2x + 2\cos^2\frac{x}{2} = 1$, where $0 < x < 2\pi$.

Example 3

Express $\sin\theta$ and $\cos\theta$ in terms of $\frac{\theta}{2}$ and deduce that

$\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ and

$\tan\theta = \frac{2t}{1-t^2}$, where $t = \tan\frac{\theta}{2}$

Detailed Content	Time Ratio	Notes on Teaching

Hence, solve the equation $\frac{3\cos\theta + 4\sin\theta}{3-5\sin\theta} = 3$

for $0° < \theta < 180°$

Example 4

Prove that for $\sin\theta \neq 0$

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = \frac{\sin 8\theta}{2\sin\theta}$$

In this example, teachers may guide students to investigate the expression $2\sin\theta(\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta)$ and remark that the summation of trigonometric series such as

$C=\cos\theta + \cos(\theta+\alpha) + \cos(\theta+2\alpha + \ldots + \cos(\theta+n\alpha)$
and
$S= \sin\theta + \sin(\theta+\alpha) + \sin(\theta+2\alpha) + \ldots + \sin(\theta+n\alpha)$
can be treated in a similar way.

Example 5

Solve the following equations for $0^\circ \le \theta \le 360^\circ$:

(*a*) $\sin\theta + \sin 3\theta + \sin 5\theta=0$

(*b*) $\cos 3\theta \cos 8\theta + \sin 4\theta \sin 7\theta=0$

Detailed Content	Time Ratio	Notes on Teaching
4.4 The Subsidiary Angle Form	~~2~~ 3	Teachers may ask students to express $r\cos(\theta-\alpha)$ in the form $a\cos\theta + b\sin\theta$. Then, students should be able to see that $a=r\cos\alpha$ and $b=r\sin\alpha$. Conversely, the expression $a\cos\theta + b\sin\theta$ can be written in the form $r\cos(\theta-\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$. Teachers may illustrate these with the following right-angled triangle:

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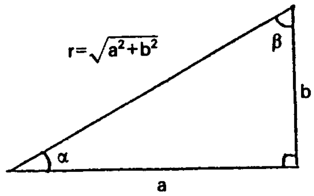
Detailed Content	Time Ratio	Notes on Teaching



Teachers should remark that the expression a cos θ + b sin θ can also be written as $r\sin(\theta+\beta)$, using the complementary angle $\beta = \frac{\pi}{2} - \alpha$. As the value of a sine or cosine function can only vary from - 1 to 1, students should have no difficulty to see that

$$-r \le r\cos(\theta-\alpha) \le r$$

and $-r \le r\sin(\theta+\alpha) \le r$.

Teachers may discuss with students the shape of the graph of the form $y = a\cos\theta + b\sin\theta$. Examples such as the following can also be introduced.

Example 1

(a) If $\sqrt{3}\sin\theta + \cos\theta = r\sin(\theta+\alpha)$ where r>0 and $0^\circ \le \alpha \le 90^\circ$, find r and α.

(b) If $y = \frac{1}{\sqrt{3}\sin\theta + \cos\theta + 7}$, using the result in *(a)*, find the range of values of y.

Example 2

Let $f(\theta) = 12\sin\theta - 5\cos\theta + 8$.

(a) Express f(θ) in the form r sin(θ – α) + C where r, α and C are constants and $0^\circ \le \theta \le 90^\circ$.

(b) Using the result in (a), or otherwise, find he least value of f(θ).

Detailed Content	Time Ratio	Notes on Teaching
4.5 General Solution of Simple Trigonometric Equations	~~4~~ 5	Teachers may ask students to give all the solutions of the equation $\sin\theta = 30^\circ$. Students should be able to give $\theta = 30^\circ$ as one possible value of θ and 180°–30°, 360°+30°, 540°–30°, … as other possible solutions.

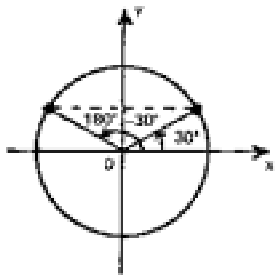
Teachers are then expected to guide students to combine all the results into two sets.

$\theta = 2n\,(180^\circ) + 30^\circ$ and

$\theta = (2n-1)(180^\circ) - 30^\circ$ and

into a single formula by writing

$\theta = n \cdot 180^\circ + (-1)^n(30^\circ)$ where n is an integer.



At this stage, students should have no difficulty to see that the general solutions of $\sin\theta = \sin\alpha$ is $\theta = n \cdot 180^\circ + (-1)^n\alpha$.

Teachers may discuss with students the general solution of $\cos\theta = \cos\alpha$ and concluded that $\theta = n \cdot 360^\circ \pm \alpha$.

Similarly, discussion on $\tan\theta = \tan\alpha$ should give $\theta = n \cdot 180^\circ + \alpha$.

It is desirable for the teachers to rewrite the above three results in terms of radian measures.

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Detailed Content	Time Ratio	Notes on Teaching

$\theta = n\pi + (-1)^n \alpha$ if $\sin\theta = \sin\alpha$

$\theta = 2n\pi \pm \alpha$ if $\cos\theta = \cos\alpha$

$\theta = n\pi + \alpha$ if $\tan\theta = \tan\alpha$

Students should be reminded not to mix radians and degrees in the same formula. They may use the inverse function notations such as $\theta = n\pi + (-1)^n \sin^{-1}(\frac{1}{2})$.

The following examples are quite typical.

Example 1

Find the general solution of $\cos^2 y = \frac{1}{2}$.

The equation is equivalent to $\cos y = \pm\frac{1}{\sqrt{2}}$. It leads to the solution $y = 2n\pi \pm \frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$, where n is an integer. For abler students, the solution can be rearranged as follows:

For $y = 2n\pi \pm \frac{\pi}{4}$,

$y = k\pi \pm \frac{\pi}{4}$, where k is an even integer or zero.

For $y = 2n\pi \pm \frac{3\pi}{4}$,

$y = k\pi \pm \frac{\pi}{4}$, where k is an odd integer.

Combining, the solution becomes $y = n\pi \pm \frac{\pi}{4}$, where n is an integer.

Example 2

Find the general solution of $2\cos\theta = \cot\theta$. Students might overlook the solution $\cos\theta = 0$. Teachers should remind them not to cancel any factor, which might be zero.

Detailed Content	Time Ratio	Notes on Teaching

Example 3

Find the general solution of $\cos 3\theta = \sin\theta$. This may be solved in terms of sine or cosine. In terms of sine, it leads to $\theta = 180°n + (-1)^n(90° - 3\theta)$ and in terms of cosine, it leads to $3\theta = 360°n \pm (90° - \theta)$. So the formula for cosine is easier to handle.

Example 4

Find the general solution of sin 2x + sin 4x = cos x.

The identity $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ may be used to rewrite the equation in the form 2 sin 3x cos x = cos x.

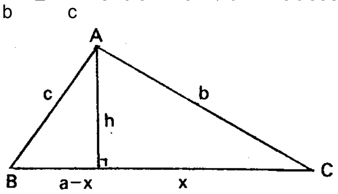
Example 5

Find the general solution of $2\sin\frac{\theta}{2}\sin\frac{3\theta}{2} = 1$. The equation may be solved by using the identity 2 sin A sin B = cos (A – B) – cos (A + B) or the identity $\sin 3A = 3\sin A - 4\sin^3 A$.

Example 6

Find the general solution of $\sqrt{3}\cos\theta + \sin\theta = 1$. Students may solve the equation by expressing $\sqrt{3}\cos\theta + \sin\theta$ in the form $r\sin(\theta + \alpha)$ or $r\cos(\theta - \alpha)$.

Detailed Content	Time Ratio	Notes on Teaching
4.6 Solution of Triangles	10*	The sine and cosine formulae are required for the solution of triangles. For any acute-angled triangle ABC, it is not difficult for students to see and derive the relations $\frac{\sin B}{b} = \frac{\sin C}{c}$ and $c^2 = a^2 + b^2 - 2ab\cos C$ from the following diagram.



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Detailed Content	Time Ratio	Notes on Teaching

The formulae can be extended and applied to any triangles, including obtuse-angled triangles.

Alternatively, the sine formula can be derived by considering any triangle and its circumcircle. Instead of using the Pythagoras' Theorem, the cosine formula can be derived from the following three identities:

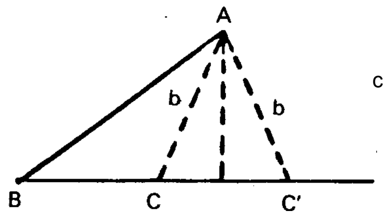
a = b cos C + c cos B
b = a cos C + c cos A
c = b cos A + a cos B

In this case students should note that the Pythagoras' Theorem is a special case of the cosine formula.

Sufficient examples and exercises should be given to students. Exercises on the solution of triangles should include cases when given:

(1) 2 angles and any side
(2) 2 sides and a non-included angle
(3) 2 sides and the included angle
(4) 3 sides

Students should be able to select and apply the formulae wherever appropriate. The ambiguous case should be clearly explained with examples. The following diagram is certainly helpful here.



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Detailed Content	Time Ratio	Notes on Teaching
4.7 Problems in Two and Three dimensions	~~4~~ 6	Teachers should introduce harder problems including those, which involve the use of compound angles and any of the six trigonometric ratios.

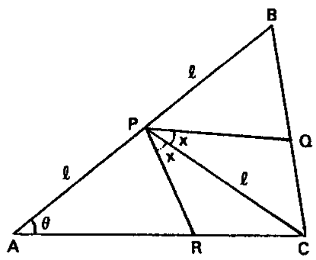
Detailed Content	Time Ratio	Notes on Teaching

Students should be encouraged to sketch the relevant diagrams before solving the problems. Wire-models or 3-dimensional teaching aids are useful for explanation and illustration.

Due amount of examples and exercises are recommended for helping students master the relevant problem solving skill.

Example 1

In Fig. 1, ABC is a triangle with $\angle A = \theta$. P is a point on AB such that PA=PB=PC= ℓ . R and Q are points on AC and BC, respectively, such that $\angle QPC = \angle RPC = x$.



(*a*) Show that $PR = \dfrac{\ell \sin\theta}{\sin(x+\theta)}$.

(*b*) Find $\angle PCQ$ in terms of θ and hence find PQ in terms of ℓ, x and θ .

(*c*) Show that the area of $\Delta PQR = \dfrac{\ell^2 \sin\theta\cos\theta\sin 2x}{2\sin(x+\theta)\cos(x-\theta)}$ and show that it can be expressed as $\dfrac{\ell^2 \sin 2\theta}{2}(1-\dfrac{\sin 2\theta}{\sin 2x + \sin 2\theta})$ ……………………..….(*)

(*d*) (i) If $\theta = \dfrac{\pi}{8}$, find the possible range of values of x.

Hence use (*) to deduce the maximum area of Δ PQR and express it in terms of ℓ .

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Detailed Content	Time Ratio	Notes on Teaching

(ii) If $\theta = \frac{\pi}{12}$, what is the possible range of values of x?

Express the maximum area of ΔPQR in terms of ℓ.

Example 2

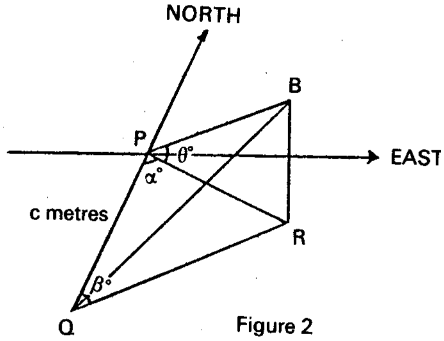

Figure 2

A balloon B is observed simultaneously from two points P and Q on a horizontal ground, P being at a distance c metres due north of Q. The bearings of the balloon from P and Q are $S\alpha°E$ and $N\beta°E$ respectively. The angle of elevation of B from P is θ°. R is the projection of B on the ground (see Figure 2).

(a) Show that the balloon is at a height h metres where

$$h = \frac{c\tan\theta°\sin\beta°}{\sin(\alpha° + \beta°)}$$

(b) Given θ=40, α=54 and β=46,

(i) find the angle of elevation of B from Q;

(ii) find the angle of elevation and the bearing of B from M, where M is the mid-point of PQ.

Time Ratio: ~~16*+24~~ 18*+28

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UNIT 5: Limits and Differentiation.

Specific Objectives:

1. To learn the concept of limits.
2. To evaluate the limits of functions.
3. To find the derivatives of functions.
4. To apply the technique of differentiation to problem solving.

Detailed Content	Time Ratio	Notes on Teaching
5.1 Limits	~~5~~ 6	See below

Teachers should review the idea and notation of a function. The limit of a function f(x) at x_0 can be introduced as the value at which f(x) would approach to as x approaches x_0. Teachers may point out that the limit of a function f(x) at $x=x_0$ is equal to the value of the function at x_0 if and only if the function is continuous at $x=x_0$, but rigorous treatment of continuity is not required. Notations such as $\lim_{x \to a} f(x)$ and $\lim_{h \to 0} f(x+h)$ should be introduced. Teachers should discuss with students the limits of simple algebraic functions and the limits of simple rational functions at infinity. Examples such as the following should be introduced to clarify the concept.

Example 1

Draw the graph of $f(x) = \frac{x^2 - 4}{x - 2}$ and consider the value of f(x) as x approaches 2.

Teachers may allow students to tabulate the following table and draw the graph of f(x):

x	0	1	1.9	1.99	1.999	2	2.001	2.01	2.1	3
f(x)	2	3	3.9	3.99	3.999	undefined	4.001	4.01	4.1	5

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