Detailed Content	Time Ratio	Notes on Teaching
		(ii) If $\theta = \frac{\pi}{12}$, what is the possible range of values of x? Express the maximum area of ΔPQR in terms of ℓ.

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Example 2

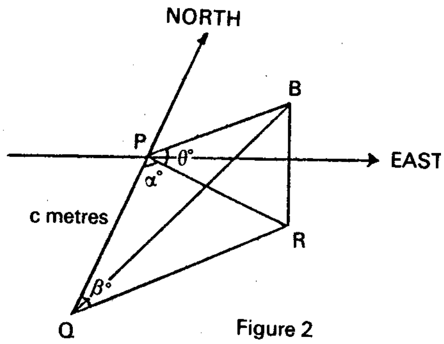

Figure 2

A balloon B is observed simultaneously from two points P and Q on a horizontal ground, P being at a distance c metres due north of Q. The bearings of the balloon from P and Q are Sα°E and Nβ°E respectively. The angle of elevation of B from P is θ°. R is the projection of B on the ground (see Figure 2).

(a) Show that the balloon is at a height h metres where

$$h = \frac{c\tan\theta^\circ \sin\beta^\circ}{\sin(\alpha^\circ + \beta^\circ)}$$

(b) Given θ=40, α=54 and β=46,
(i) find the angle of elevation of B from Q;
(ii) find the angle of elevation and the bearing of B from M, where M is the mid-point of PQ.

Time Ratio: ~~16*+24~~ 18*+28

UNIT 5: Limits and Differentiation.

Specific Objectives:

1. To learn the concept of limits.
2. To evaluate the limits of functions.
3. To find the derivatives of functions.
4. To apply the technique of differentiation to problem solving.

Detailed Content	Time Ratio	Notes on Teaching
5.1 Limits	~~5~~ 6	Teachers should review the idea and notation of a function. The limit of a function f(x) at x_0 can be introduced as the value at which f(x) would approach to as x approaches x_0. Teachers may point out that the limit of a function f(x) at $x=x_0$ is equal to the value of the function at x_0 if and only if the function is continuous at $x=x_0$, but rigorous treatment of continuity is not required. Notations such as $\lim_{x\to a} f(x)$ and $\lim_{h\to 0} f(x+h)$ should be introduced. Teachers should discuss with students the limits of simple algebraic functions and the limits of simple rational functions at infinity. Examples such as the following should be introduced to clarify the concept.

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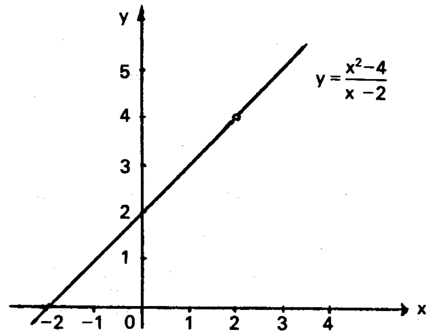
Example 1

Draw the graph of $f(x) = \frac{x^2 - 4}{x - 2}$ and consider the value of f(x) as x approaches 2.

Teachers may allow students to tabulate the following table and draw the graph of f(x):

x	0	1	1.9	1.99	1.999	2	2.001	2.01	2.1	3
f(x)	2	3	3.9	3.99	3.999	undefined	4.001	4.01	4.1	5

Detailed Content	Time Ratio	Notes on Teaching



When x approaches 2, f(x) approaches the value 4 and this is denoted by

$$\lim_{x \to 2} f(x) = 4$$

or

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

It is obvious that the graph of $f(x) = \frac{x^2 - 4}{x - 2}$ is the same as y=x+2 except for the point (2,4). When x approaches the value 2 from either side, we may write $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x+2) = 4$

However, teachers should emphasize that f(x) is not defined at x=2.

Example 2

Evaluate $\lim_{x \to 2} x^2$

Detailed Content	Time Ratio	Notes on Teaching

As the function $f(x)=x^2$ is continuous at x=2, the limit is equal to f(2), so

$$\lim_{x \to 2} x^2 = 2^2 = 4$$

Teachers may illustrate the case graphically and numerically.

Example 3

Evaluate $\lim_{x \to 8} \frac{x - 8}{\sqrt{x+1} - 3}$

Teachers should remind students of the technique of rationalization:

$$\frac{x-8}{\sqrt{x+1}-3} = \frac{(x-8)(\sqrt{x+1}+3)}{(\sqrt{x+1}-3)(\sqrt{x+1}+3)} = \frac{(x-8)(\sqrt{x+1}+3)}{(x+1)-9}$$

$$= \sqrt{x+1} + 3$$

Hence, $\lim_{x \to 8} \frac{x-8}{\sqrt{x+1}-3} = \lim_{x \to 8} (\sqrt{x+1} + 3) = 6$

Again, students should note that $\frac{x-8}{\sqrt{x+1}-3}$ is not defined at x=8.

Example 4

Consider the limits of $f(x) = \frac{1}{x}$ as x approaches infinity. Teachers may introduce the notation $\lim_{x \to \infty} f(x)$.

Students should be able to tabulate the following table and draw the graph of $\frac{1}{x}$:

x	−100	−10	−1	−0.1	0	0.1	1	10	100	1000
f(x)	−0.01	−0.1	−1	−10	undefined	10	1	0.1	0.01	0.001

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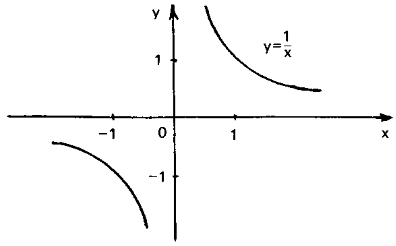
Detailed Content	Time Ratio	Notes on Teaching



Students should be able to see that as x approaches infinity, $\frac{1}{x}$ approaches 0,

i.e. $\lim_{x\to\infty} = \frac{1}{x} = 0$. Teachers may also show that $\lim_{x\to-\infty} = \frac{1}{x} = 0$.

When evaluating limits of functions, the following theorems may be used:

If $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist, then

(*a*) $\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$

(*b*) $\lim_{x\to x_0} [f(x)\cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$

(*c*) $\lim_{x\to x_0} [c \cdot f(x)] = c \cdot \lim_{x\to x_0} f(x)$, where c is a constant.

(*d*) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ provided that $\lim_{x\to x_0} g(x) \neq 0$.

Rigorous proofs of the above theorems are not expected.

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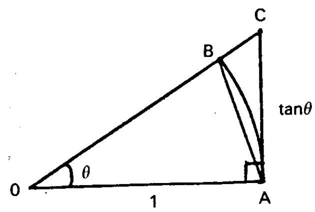
Detailed Content	Time Ratio	Notes on Teaching

When dealing with trigonometric functions, the limit $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, where θ is in radians, is useful. Teachers can guide students to arrive at the result by comparing areas of ΔAOB , sector AOB and Δ AOC as shown in the figure.



Teachers may also encourage students to use calculators to verify $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ by evaluating the values of $\frac{\sin\theta}{\theta}$ for decreasing values of θ such as 0.1, 0.01, 0.001 and 0.0001.

Examples and exercises may include $\frac{\sin 3\theta}{2\theta}, \frac{\tan 4\theta}{2\theta}, \frac{1-\cos\theta}{\theta^2}$ and $\frac{\sin m\theta}{\sin n\theta}$ where m, n are constants.

Detailed Content	Time Ratio	Notes on Teaching
5.2 Derivatives	~~4~~ 5	Using notations such as Δx and Δy to represent increments, the derivative of a function y=f(x) with respect to x may be defined as $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ provided that the limit exists. Common notations of derivatives such as f'(x), $\frac{dy}{dx}$, y', $\frac{d[f(x)]}{dx}$ and $\frac{d}{dx}[f(x)]$ should be introduced.

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Detailed Content	Time Ratio	Notes on Teaching
		Teachers should emphasize that $\frac{dy}{dx}$ is a symbol and should not be taken as a fraction. Examples should be given to show how to obtain the derivatives from first principles. Examples should include simple polynomials and algebraic expressions of the form $\frac{1}{ax+b}$, $\frac{ax+b}{cx+d}$, $\sqrt{ax+b}$, and $\frac{ax+b}{\sqrt{cx+d}}$ where a, b, c, d are constants. The treatment on trigonometric functions may be discussed after introducing the rules of differentiation.
5.3 Differentiation **5.3.1 Simple Algebraic Functions and Rules of Differentiation**	~~4~~ 5	Teachers may derive the following rules of differentiation where c is a constant and u, v are functions of x. 1. $\frac{dc}{dx}=0$ 2. $\frac{d}{dx}[x^n]=nx^{n-1}$ where n is an integer. (Power Rule) Note: The Power Rule $\frac{d}{dx}[x^n]=nx^{n-1}$ for n being a positive integer may be introduced and used without proof. The case when n is an integer (positive and negative) may be discussed after the Quotient Rule. 3. $\frac{d}{dx}[cf(x)]=c\cdot\frac{d}{dx}[f(x)]$ 4. $\frac{d}{dx}[u+v]=\frac{du}{dx}+\frac{dv}{dx}$ 5. $\frac{d}{dx}[u\cdot v]=u\frac{dv}{dx}+v\frac{du}{dx}$ (Product Rule) 6. $\frac{d}{dx}[\frac{u}{v}]=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ (Quotient Rule)

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Detailed Content	Time Ratio	Notes on Teaching
		It is desirable to begin with polynomials as examples. Teachers should show how to differentiate a constant, powers of x and linear combinations of the powers of x. When the students are familiar with the Power Rule $\frac{d}{dx}[x^n]=nx^{n-1}$ for n=0, 1, 2….., they should be able to differentiate a polynomial term by term. When rules for the differentiation of a sum, a product and a quotient of functions are established, students should be able to differentiate the products of polynomials and rational functions such as $(2x+3)(4x^2+5)$ and $\frac{1-2x^2}{2+3x}$. The following examples are also useful: *Example 1* Find the derivative of x^n, where n is a positive integer, from first principles. *Example 2* Given that $\frac{d[x^n]}{dx}=nx^{n-1}$ for n being a positive integer, and that $\frac{d}{dx}[\frac{1}{f(x)}]=\frac{-\frac{d}{dx}[f(x)]}{[f(x)]^2}$ for $f(x)\neq 0$, show that $\frac{d[x^n]}{dx}=nx^{n-1}$ is true for all integers.
5.3.2 Differentiation of Composite Functions and Implicit Functions	~~4~~ 6	Teachers should give some typical examples of composite functions, implicit functions and inverse functions. Teachers should then introduce the Chain Rule: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ where y is a function of u and u is a function of x. The Chain Rule may be used for the differentiation of composite functions, implicit functions and inverse functions. Teachers should introduce two more useful formulae:

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Detailed Content	Time Ratio	Notes on Teaching

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$$

and $$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Rigorous proofs of these formulae are not required but sufficient examples on their applications are necessary. Teachers may use the Chain Rule to show that the Power Rule $\frac{d}{dx}[x^n] = nx^{n-1}$ is also valid for n being a rational number. By now students should be able to apply the rules of differentiation to algebraic expressions involving rational indices such as $(3x^2+4)^{\frac{5}{2}}$ or $(2x-3)^{-\frac{3}{2}}$. Teachers may use the equations of ~~conic sections~~ curves in standard form or parametric form in examples and exercises.

Example 1

Differentiate $y = (2x^2+1)^5$ with respect to x. With some practice, students may be able to apply the Chain Rule and present the following without making the substitution $u=2x^2+1$:

$$\frac{dy}{dx} = \frac{d(2x^2+1)^5}{d(2x^2+1)} \cdot \frac{d(2x^2+1)}{dx}$$

$$= 5(2x^2+1)^4 \cdot 4x$$

$$= 20x(2x^2+1)^4$$

Example 2

Find $\frac{dy}{dx}$ given the following parametric equations:

Detailed Content	Time Ratio	Notes on Teaching

$$\begin{cases} x = t^2 \\ y = 2t \end{cases}$$

Students should be able to make use of the fact

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$

in their work.

Example 3

Find the value of $\frac{dy}{dx}$ of the function $x^2 + y^2 = 25$ at the point (3,4). Teachers may illustrate the result geometrically.

Example 4

Given that $\frac{d}{dx}[x^n] = nx^{n-1}$ is true for any integer n. Show that $\frac{d}{dx}[x^n] = nx^{n-1}$ is also true for $n = \frac{p}{q}$ where p, q are integers and q>0.

Let $y = x^n = x^{\frac{p}{q}}$, then $y^q = x^p$.

Differentiating implicitly with respect to x,

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

or $$\frac{dy}{dx} = (\frac{p}{q}) \cdot \frac{x^{p-1}}{y^{q-1}}$$

$$= (\frac{p}{q}) \cdot x^{\frac{p}{q}-1}$$

showing that the power rule is valid for n being a rational number.

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Detailed Content	Time Ratio	Notes on Teaching
5.3.3 Differentiation of Trigonometric Functions	~~3~~ 5	Teachers might have to review some of the limits of trigonometric functions and some of the compound angles formulae or sum and product formulae before deriving the derivatives of the six trigonometric functions. Students should also be reminded frequently that all angles must be in radian measures. The derivative of sin x may be derived as follows: $\frac{d}{dx}[\sin x] = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$ $= \lim_{h\to 0}\frac{2\cos(x+\frac{h}{2})\cdot\sin(\frac{h}{2})}{h}$ $= \lim_{h\to 0}\cos(x+\frac{h}{2})\cdot\lim_{h\to 0}\frac{\sin(\frac{h}{2})}{\frac{h}{2}}$ $= \cos x \cdot 1$ $= \cos x$ The derivatives of other functions may be derived from first principles or by applying the above formula and rules of differentiation. The following shows how cos x can be differentiated. $\frac{d}{dx}[\cos x] = \frac{d}{dx}[\sin(\frac{\pi}{2}-x)]$ $= -\cos(\frac{\pi}{2}-x)$ $= -\sin x$ In fact, students may be asked to differentiate tan x, cot x, sec x and cosec x themselves to arrive at the results:

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Detailed Content	Time Ratio	Notes on Teaching
		$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$ Teachers should provide ample examples and exercises to help students apply these rules together with the other rules of differentiation to functions composed of simple algebraic and trigonometric functions. However, differentiation of inverse trigonometric functions is unnecessary.
5.3.4 Second Derivative ~~Higher Derivatives~~	~~3~~ 2	The second derivative ~~Higher derivatives~~ may be obtained by differentiating ~~successive differentiation of~~ the first derivative. Given a function y=f(x) which may be differentiated to give its first derivative $y' = \frac{dy}{dx} = f'(x)$, the derivative of f′(x) is called the second derivative and is denoted by $y'' = \frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx}[\frac{dy}{dx}]$. The n^{th} derivative may be defined inductively and teachers should introduce its conventional notations: $f^{(n)}(x) = \frac{d^n y}{dx^n} = y^{(n)} = \frac{d}{dx}[\frac{d^{n-1}y}{dx^{n-1}}]$ The following examples can be introduced: *Example 1* Given x = a sin (wt + k) where a, w and k are constants, show that $\frac{d^2x}{dt^2} = -w^2x$. *Example 2* Let $f(\theta) = \sqrt{\theta^2 + k}\,\sin 2\theta$, where k is a constant and $f'(0) = 1$, find the value of k.

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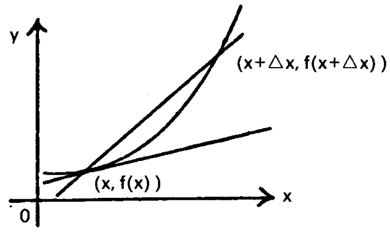
Detailed Content	Time Ratio	Notes on Teaching
		Example 3
		If y = a sin x + cos x satisfies the equation $\frac{d^2y}{dx^2} + y = b$ where a, b are constants and when $t = 0$, $\frac{dy}{dx} = 2$, find the values of a and b.
5.4 Applications of Differentiation **5.4.1 Gradients, Tangents and Normals to a Curve**	~~4~~ 5	The difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta y}$ can be interpreted as the gradient of a chord joining the points (x, f(x)) and $(x + \Delta x, f(x + \Delta x))$ on the graph of the function y=f(x). As Δ x approaches 0, the limiting value of $\frac{\Delta y}{\Delta x}$ will give the slope or gradient of the tangent to the curve at the point (x, f(x)).



Teachers may illustrate the fact by drawing tangents to a simple graph such as $y = x^2, y = \frac{1}{x}$ or $x^2 + y^2 = 25$ as shown in the following examples.

Example 1

Consider the tangent to the curve $y = x^2$ at $x = 2$.

A well-drawn tangent to the curve at x=2 will have gradient equal to 4. Teacher should remind students that the value of the derived function at x=2 can be evaluated from first principles as follows:

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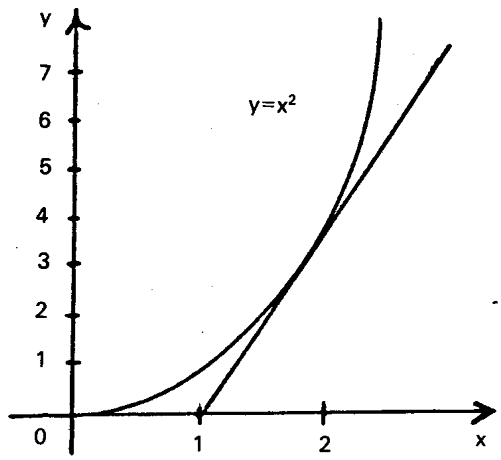
Detailed Content	Time Ratio	Notes on Teaching
		$\left[\frac{dy}{dx}\right]_{x=2} = \lim_{x\to 2} \frac{x^2 - 2^2}{x - 2} = 4$ in which the difference quotient $\frac{x^2 - 2^2}{x - 2}$ is simply the slope of a line joining the points (2,4) and a variable point (x, x^2) on the curve y=x^2.



Example 2

Consider the tangents to the graph of $y = \frac{1}{x}$. The slope of the tangent to the curve $y = \frac{1}{x}$ at x=2 (say) can be easily read from the graph. The result can be verified by differentiation:

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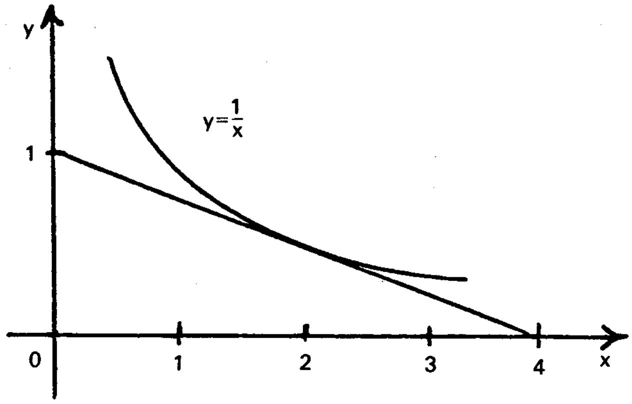
Detailed Content	Time Ratio	Notes on Teaching
		$\left(\frac{dy}{dx}\right)_{x=2} = -\frac{1}{x^2}\Big\vert_{x=2} = -\frac{1}{4}$ 
5.4.2 Maxima and Minima Sketching of Simple Curves	~~5~~ 7	Once students are convinced of the fact, they should be able to find tangents and normals of simple curves. In this syllabus, both relative extrema and absolute extrema (greatest/least value) should be considered. Students are also expected to distinguish between relative and absolute extrema. Students should be guided to acquire the facts: (i) If $f'(x_1)>0$, then the function is increasing at x_1; (ii) If $f'(x_1)<0$, then the function is decreasing at x_1; (iii) If $f'(x_1)=0$, then the function has a stationary point at x_1.

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Detailed Content	Time Ratio	Notes on Teaching
		Students should be clear that if $f'(x)=0$ and $f'(x)$ changes from positive to negative as x increases through x_1, then f(x) attains a relative maximum at x_1. If $f'(x_1)=0$ and $f'(x)$ changes from negative to positive as x increases through x_1, then f(x) attains a relative minimum at x_1. If the second derivative is not zero at x_1, it can also be used to determine whether $f(x_1)$ is a relative maximum or a relative minimum. If $f'(x_1)=0$ and $f''(x_1)<0$, then f(x) attains a relative maximum at x_1. If $f'(x_1)=0$ and $f''(x_1)>0$, then f(x) attains a relative minimum at x_1. Teachers should emphasize that both the first derivative and second derivative can be used to test the extremium points, but the former is preferred if the first derivative is too complicated to differentiate. Students should note that in case the domain includes an end point, the function may attain an extreme value at this point. The first derivative of a function provides a means of finding the turning points (maximum/minimum) of the function and the intervals in which the function is increasing or decreasing. The shape of the curve of the function can be determined by plotting only a few critical points. For example, to sketch the curve $y=(x-1)^2(x+1)$ for $-2\le x\le 2$, teachers should guide students to find where the curve crosses the axes and the values of x at which $\frac{dy}{dx}$ is negative, zero or positive. After getting all the information, students should have no difficulty in sketching the curve.

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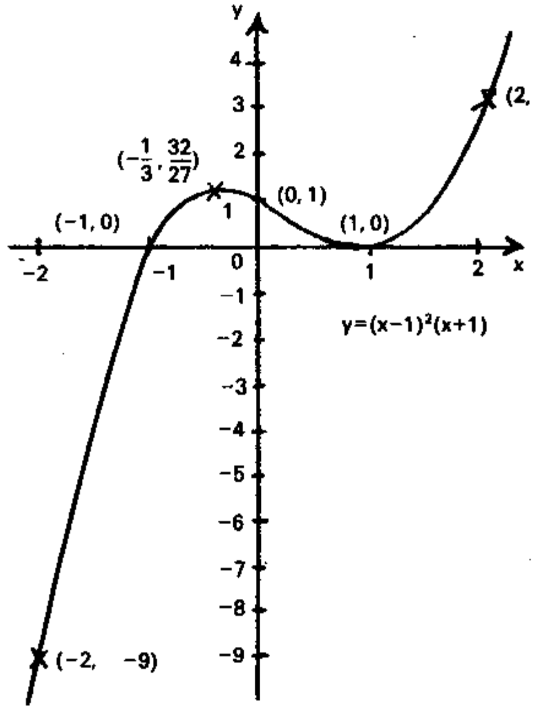
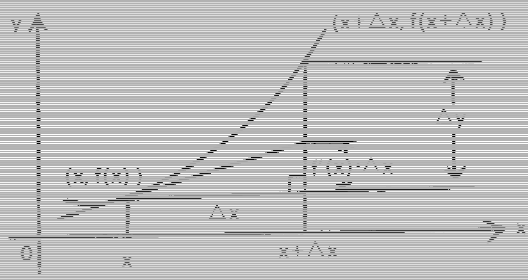
Detailed Content	Time Ratio	Notes on Teaching
		 Teachers should remind students to label their sketches properly. Sketching of the curves such as $y=\frac{x^2-2x+1}{x^2+2}$ and $y=\frac{4x-2}{x^2+4}$ are also required. For abler students teachers may discuss points of inflexion and asymptotes.

Detailed Content	Time Ratio	Notes on Teaching
5.4.3 Rates of Change	~~2~~ 3	If x is a function of time t, then the derivative $\frac{dx}{dt}$ will give the rate of change of x with respect to time t. Velocity and acceleration are good examples of rates of change. Problems involving related rates of change should also be discussed. Two of them are shown below. *Example 1* The base radius of an inverted conical funnel is 30 cm and the height is 40 cm. Water is running out of it at the bottom at the rate of πcm^3s^{-1}. Find the rate at which the water level is falling when the depth of the water is 20 cm. *Example 2* A man 2 m tall is walking along a straight road away at $2ms^{-1}$ from a lamp 6 m high. Find the rate of increase of the length of his shadow.
55 5.4.4 Small Increments	2	If y=f(x) and Δx is a sufficiently small increment of x, it can be seen that $\Delta y \approx f'(x)\Delta x$ and $f(x+\Delta x) = f(x)+\Delta y \approx f(x)+f'(x)\Delta x$  Teachers should show students how to apply the idea to approximation and estimation of errors.

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Detailed Content	Time Ratio	Notes on Teaching
		Example 1 Given that $\cos 60^{\circ} = \frac{1}{2}$, find an approximate value of $\cos 61^{\circ}$ by using the method of small increments. Let $f(x) = \cos x$, $f'(x) = -\sin x$ $x = 60^{\circ} = \frac{\pi}{3}$ radians $\delta x = 1^{\circ} = \frac{\pi}{180}$ radian $\cos 61^{\circ} = \cos(\frac{\pi}{3} + \frac{\pi}{180})$ $\approx \cos(\frac{\pi}{3}) + (-\sin\frac{\pi}{3})(\frac{\pi}{180})$ $= 0.5 - \frac{\sqrt{3}}{2}(\frac{\pi}{180})$ $\therefore \cos 61^{\circ} \approx 0.485$ Teachers should remind students that the variable x in the function f(x)=cos x should be in radians. *Example 2* If the radius of a solid sphere is measured with a percentage error of 0.1% and the volume is calculated. Estimate the percentage error of the volume.
	~~36~~ 44	

UNIT 6: Inequalities

Specific Objectives:

1. To understand the basic rules of inequalities.
2. To solve linear inequalities in one variable.
3. To solve quadratic inequalities in one variable.
4. To solve inequalities of the form $\frac{ax+b}{cx+d} \geq k$.
5. To solve linear and quadratic inequalities involving absolute values.
6. To prove simple absolute inequalities.

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Detailed Content	Time Ratio	Notes on Teaching
6.1 Basic Rules of Inequalities	1*	Teachers should emphasize that if a–b is a positive number then a>b, and vice versa. With this fact, the following basic rules can be derived. For real numbers a, b, c: (1) If a>b, and b>c, then a>c. (2) If a>b, then a+c>b+c (3) If a>b, then (a) ac>bc for c>0. (b) ac<bc for c<0. (c) ac=bc for c=0. Simple proofs of inequalities by using the basic rules should be introduced.
6.2 Linear Inequalities in one Variable	~~1~~ 1*+1	Students should be reminded that the method of solving linear inequalities resembles that of solving linear equations. The only difference is that when an inequality is multiplied or divided by a negative number, then the inequality sign has to be reversed.

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