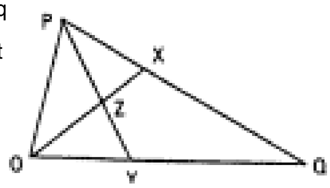
90

Detailed Content	Time Ratio	Notes on Teaching
		(3) An angle in a semi-circle is 90^0. (4) The diagonals of a rhombus bisect each other perpendicularly. (5) The altitudes of a triangle are concurrent. (6) The medians of a triangle are concurrent. Other examples may include the following. *Example 1* In the figure, $\overrightarrow{OP}=\vec{p}, \overrightarrow{OQ}=\vec{q}, 2\overrightarrow{PX}=\overrightarrow{XQ}$ and $3\overrightarrow{OY}=2\overrightarrow{YQ}$. (*a*) Express $\overrightarrow{OX}$ in terms of $\vec{p}$ and $\vec{q}$ (*b*) If $\overrightarrow{OZ}=\lambda\overrightarrow{OX}, \overrightarrow{PZ}=\mu\overrightarrow{PY}$, show that $5\lambda-6\mu=0$ and $2\lambda+3\mu-3=0$. (*c*) By solving for λ, find PZ:PY.  *Example 2* Show that if $\vec{u}$ and $\vec{v}$ are any vectors, then (*a*) $\lvert\vec{u}\cdot\vec{v}\rvert \le \lvert\vec{u}\rvert\lvert\vec{v}\rvert$ (*b*) $\lvert\vec{u}+\vec{v}\rvert \le \lvert\vec{u}\rvert+\lvert\vec{v}\rvert$.
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91

UNIT 9: Integration

Specific Objectives:

1. To recognize indefinite integration as a reverse process of differentiation.
2. To understand the properties of indefinite integrals.
3. To recognize some of the geometric and physical applications of indefinite integration.
4. To recognize and use some standard formulae of indefinite integration and the method of substitution.
5. To understand the underlying principle of definite integral as a limit of a sum.
6. To understand and apply the basic properties of definite integration.
7. To apply definite integration to find plane areas and volumes of solids of revolution.

Detailed Content	Time Ratio	Notes on Teaching
9.1 Indefinite Integral	~~1~~ 2	Indefinite Integration is introduced as the reverse process of differentiation, which leads to the simple properties. $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$ and $\int ku\,dx = k\int u\,dx$. The terms primitive function, integral sign, integrand and constant of integration should be introduced. Teachers should emphasize that the integral $\int u\,dx$ is only a notation and it doesn't mean that u is multiplied by dx.
9.2 Integration of Simple Functions and Simple Applications	~~4~~ 5	Ample examples and exercises should be provided to help students acquaint with the integration formulae; $\int x^n dx = \frac{1}{n+1}x^{n+1} + C (n \neq -1)$ (The case when n=0 should be discussed.) $\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int \sec^2 x\,dx = \tan x + C$ $\int \sec x \tan x\,dx = \sec x + C$

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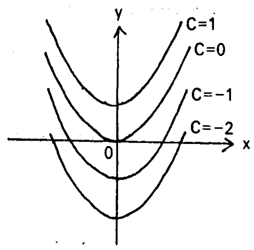
92

Detailed Content	Time Ratio	Notes on Teaching
		$\int \text{cosec}^2 x\,dx = -\cot x + C$ $\int \text{cosec}\,x \cot x\,dx = -\text{cosec}\,x + C$ Integration of $\frac{1}{x}$ and the exponential function e^x is excluded. Teachers should emphasize that very often, it is necessary to modify the integrand before doing the integration. For example, $\int \sqrt{x}(2-x)\,dx = \int (2\sqrt{x} - x^{\frac{3}{2}})\,dx = \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$. $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$ $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$, etc. As geometric applications, students should appreciate that while the slope of a curve at any point in its domain determines its shape, the constant of integration determines its position relative to the coordinate axes. Thus, 2 primitive functions of the same function can differ at most by a constant. In fact, the function y=f(x)+C represents a family of curves while y=f(x) is only a particular curve in the family. The following diagram shows an example  A family of curves $y=x^2+C$

93

Detailed Content	Time Ratio	Notes on Teaching
		As physical applications, the formulae $s = \int v\,dt$ and $v = \int a\,dt$ are included. The following shows an example. *Example* A particle moves along a straight line with an acceleration a ms^{-2} given by a=4t−3 where t s is the time after passing through a point 0. When t=3, the particle has a velocity of $12ms^{-1}$. Find the velocity of the particle when t=4 and the corresponding distance travelled in the 4 seconds. In this example, teachers should guide students to observe the two initial conditions: v=12 when t=3 and s=0 (at 0) when t=0.
9.3 Simple Techniques of Integration	~~9~~ 3	(1) *The Method of Substitution* Teachers are recommended to emphasize the fact that if $F=\int f(x)dx$ is a function of x and x is a function of u, then F may be regarded transitively as a function of u and hence be differentiated with respect to u. By the chain rule $\frac{dF}{du} = \frac{dF}{dx}\cdot\frac{dx}{du} = f(g(u))g'(u)$ so that F, when expressed as a function of u, is $\int f(g(u))g'(u)\,du$. Extensive examples aiming at variety are useful. *Example 1* $\int \frac{1}{(2x-7)^2}dx = \int \frac{1}{u^2}\cdot\frac{1}{2}du$ where u=2x-7 $= \frac{-1}{2u} + C$ $= \frac{-1}{2x-7} + C$ applying $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C , n \neq -1$

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Detailed Content	Time Ratio	Notes on Teaching

94

Integration by subsitution is not required. Students are only expected to know the following formulae:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \text{ , } n \neq -1$$

$$\int \sin(ax+b)dx = -\frac{\cos(ax+b)}{a} + C$$

$$\int \cos(ax+b)dx = -\frac{\sin(ax+b)}{a} + C$$

The proof of the above formulae can be verified by differentiation.

Example 2

$\int \cos 4x$ $= \int \cos u \cdot \frac{1}{4} du$ where u=4x

$= \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin 4x + C$

applying $\int \cos(ax+b)dx = -\frac{\sin(ax+b)}{a} + C$

After mastering the technique, teachers may encourage students to omit the intermediate step of writing variable u explicitly. Therefore, the above two examples can be done as follows.

$$\int \frac{1}{(2x-7)^2}dx = \frac{1}{2}\int \frac{1}{(2x-7)^2}d(2x-7) = \frac{-1}{2x-7} + C$$

$$\int \cos 4x \, dx = \frac{1}{4}\int \cos 4x \, d(4x) = \frac{1}{4}\sin 4x + C$$

Example 3

For integrands in the form like $\frac{x^2-2x}{x^2-2x+1}$, teachers may guide students to proceed as follows:

$$\int \frac{x^2-2x}{x^2-2x+1}dx = \int \frac{(x^2-2x+1)-1}{x^2-2x+1}dx = \int \left[1 - \frac{1}{(x-1)^2}\right]dx = 1 + \frac{1}{x-1} + C$$

applying $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, $n \neq -1$

Students are not required to know the general method to resolve this type of function into partial fractions. However, given its expanded form, they should be able to find the constants and then solve for the indefinite integral.

(2) *Integration of Trigonometric Functions*

(a) Integrals of the form $\int \sin mx \cos nx dx$, $\int \cos mx \cos nx dx$ and $\int \sin mx \sin nx \, dx$

Detailed Content	Time Ratio	Notes on Teaching

95

Teachers should fully revised with students trigonometric identities like half-angle formulae sum and product formulae before working on this type of integrals. ~~Examples~~ An example like $\int \sin 4x \cos 6x \, dx$ and $\int \cos^3 x \, dx$ ~~are~~ is typical.

(b) Simple integrals of the form $\int \sin^m x \cos^n x \, dx$, $\int \tan^m x \sec^n x \, dx$ and $\int \cot^m x \csc^n x \, dx$ without using the method of substitution.

Students are expected to be familiar with the various trigonometric identities and the substitution method. ~~Typical examples include~~ A typical example is $\int \sin^2 x \cos^2 x \, dx$, $\int \cos^4 2x \, dx$, $\int \tan^3 x \sec^2 x \, dx$, $\int \sec^4 \frac{1}{2}x \, dx$, $\int \csc^3 x \cot^3 x \, dx$ etc.

(c) For other trigonometric integrands, say $\frac{1}{a+b\cos x}$, the substitution $t = \tan\frac{x}{2}$ should be introduced as a general method.

Example

To evaluate $\int \frac{1}{1+\cos\theta}d\theta$ by using the substitution $t = \tan\frac{\theta}{2}$, teachers may guide students to deduce that

$\cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2dt}{1+t^2}$.

By direct substitution, students should have no difficulty to get

$$\int \frac{1}{1+\cos\theta}d\theta = \int dt = t + C = \tan\frac{\theta}{2} + C$$

(3) *Trigonometric Substitution*

It is usually used to remove the $\sqrt{\ }$ sign in expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$. Students should know that the respective substitution are x=a sinθ (or a cosθ) x=a tanθ and x=a secθ. Ample examples and exercises should be provided to help students to get acquainted with the technique. Typical examples are $\int \sqrt{1-x^2}\,dx$,

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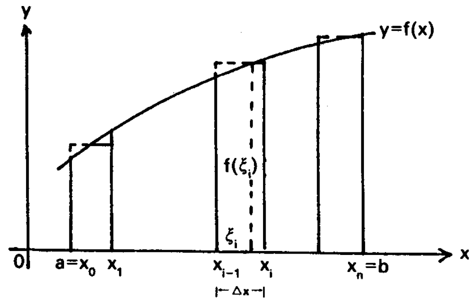
Detailed Content	Time Ratio	Notes on Teaching
		$\int \frac{\sqrt{4-x^2}}{x^2}dx$, $\int \frac{1}{x^3\sqrt{x^2-1}}dx$ and $\int \frac{2x^3}{\sqrt{4+x^2}}dx$. (4) *Simple Reduction Formulae* Integration by parts is not needed. Yet students should be given examples of simple reduction formulae derived as a reverse process of differentiation. The following example shows how this can be done. *Example* (*a*) Show that $\frac{d}{dx}(\sin^{n-1}x\cos x) = (n-1)\sin^{n-2}x - n\sin^n x$ (*b*) Let I_n denote the integral $\int \sin^n x\,dx$, show that $I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$ (*c*) Hence evaluate I_3 and I_4. Since integration is a very important process in Calculus, adequate practice is necessary in order to ensure that students can master the technique.
9.4 Definite Integral This Sub-unit is based on the techniques learned in Sub-units 9.1 to 9.3	2	With the following figure and using the idea of area, the definite integral $\int_a^b f(x)\,dx$ can be defined as the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, i.e. $\int_a^b f(x)dx = \lim_{\Delta x\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$

Detailed Content	Time Ratio	Notes on Teaching
		 The simple properties of definite integrals should be clearly stated. (1) $\int_b^a f(x)dx = 0$ (2) $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$ (3) $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ (4) $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$ (5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (6) $\int_a^b f(x)dx = \int_a^b f(u)du$ Proof of these properties are not required, though all follow easily from the definition. For abler students, the property: If f is continuous and non-negative on [a,b], then $\int_a^b f(x)dx \geq 0$ .Equally holds only when f(x)=0 for all x in [a,b], which leads to the corollary

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Detailed Content	Time Ratio	Notes on Teaching
		$f(x) \geq g(x)$ implies $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ may be discussed.
9.5 Evaluation of Definite Integral This Sub-unit is restricted to the techniques learned in Sub-units 9.1 to 9.3	~~4~~ 3	Teachers may introduce $\int_a^x f(t)dt$ as the area bounded by a positive continuous function y=f(t), the t-axis and the vertical lines t=a and t=x. Then, the formula $\int_a^x f(t)dt = [F(x)]_a^x = F(x) - F(a)$ where $\frac{d}{dx}[F(x)] = f(x)$ can be established. Examples should be provided to help students understand the formula. *Example 1* $\int_1^2 (x^2 - 1)dx = \left[\frac{x^3}{3} - x\right]_1^2 = (\frac{2^3}{3} - 2) - (\frac{1^3}{3} - 1) = \frac{4}{3}$ *Example 2* $\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$ where $x = \sin\theta$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)d\theta$ $= \frac{1}{2}[\theta + \frac{1}{2}\sin 2\theta]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{4}$ Students should be reminded of the method of substitution all the time as it is usually used in the evaluation of definite integrals (as in Example 2 above). The corresponding changes in upper and lower limit of integration should be made at the same time. Regarding ranges, teachers may like to point out the property $\int_{\frac{\pi}{2}}^{\pi} \sqrt{\cos^2 x}\,dx = \int_{\frac{\pi}{2}}^{\pi} (-\cos x)dx$ which is often overlooked by students. Integrands with special properties can also be introduced as challenging examples

98

Detailed Content	Time Ratio	Notes on Teaching
		Example 3 $\sin^5 x \cos x$ is called an odd function as it satisfies the property f(–x)= –f(x). Hence, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x \cos x dx$ is given by $\int_0^{\frac{\pi}{2}} \sin^5 x \cos x dx + \int_{-\frac{\pi}{2}}^{0} \sin^5 x \cos x dx$ $= \int_0^{\frac{\pi}{2}} \sin^5 x \cos x dx - \int_0^{\frac{\pi}{2}} \sin^5 x \cos x dx$ $= 0$ Integration of even functions can be similarly introduced through examples. *Example 4* For integrals like $\int_0^{\pi} x \sin x dx$, teachers may firstly guide students to prove the properties that if f(a+b x)=f(x), then $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]dx$ and $\int_a^b xf(x)dx = \frac{a+b}{2}\int_a^b f(x)dx$ The integral $\int_0^{\pi} x \sin x dx$ can then be evaluated by substituting a=0, b=π and f(x)=sin x.
9.6 Applications of Definite Integrals	~~8~~ 7	The formula $A = \int_a^b ydx$ or $A = \int_c^d xdy$ follows from Section 9.4. The following two figures are helpful for memorizing the above two formulae.

99

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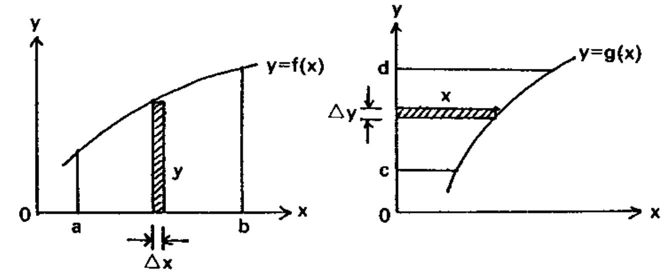
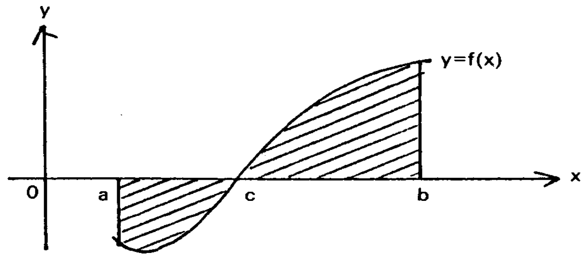
Detailed Content	Time Ratio	Notes on Teaching



The problems involved in finding volumes are restricted to solids of revolution about the x-axis or y-axis.

Teachers should lead students through examples to discover what happens when part of the curve y=f(x) lies below the x-axis as shown in the following figure. In this case, $\int_a^b f(x)dx$ only gives an algebraic sum of the areas from a to b.



To avoid misleading results, students should be encouraged to sketch the relevant curve before calculating the area. Moreover, the area should be broken up into two parts (referring to the above figure), and f(x) is integrated from a to c and from c to b. The absolute values of the two results are then added to give the required answer. The following example may be used as an illustration.

Detailed Content	Time Ratio	Notes on Teaching

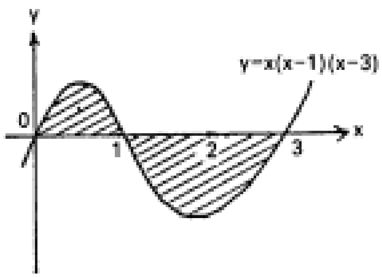
Example

Find the area bounded by the curve by $y = x(x-1)(x-3)$ and the x-axis.

In this example, students should be asked to sketch the curve themselves. Then,

$$\int_0^1 x(x-1)(x-3)dx = \frac{5}{12}$$

$$\int_1^3 x(x-1)(x-3)dx = -\frac{8}{3}$$



The required area is thus given by

$$\frac{5}{12}+\frac{8}{3}=\frac{37}{12}$$

For formulae $A=\int_a^b (y_1-y_2)dx$ and $A=\int_c^d (x_1-x_2)dy$, the approach

$\lim_{\Delta xx\to 0}\sum_{i=1}^{n}(y_{1i}-y_{2i})\Delta x = \lim_{\Delta x\to 0}\sum_{i=1}^{n} g(x_1)\Delta x = \int_a^b g(x)dx$ is recommended as it reminds students of the relation between limit of a sum and a definite integral.

Examples like the area bounded $y^2 = 4x$ and $y = 2x-4$ and that bounded by $y = \sin x$ and $y = \cos x$ where $0 \le x \le 2\pi$ may be used for demonstration.

The formula $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$ and the case when both x and y are defined in terms of a third parameter t, are not necessary.

The volumes of solids and hollow solids of revolution about the coordinate axes and lines parallel to the coordinate axes are dealt with similarly. The disc method is required and the cylindrical shell method may also be introduced for abler students in some cases. The following is an example.

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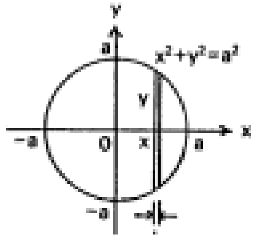
102

Detailed Content	Time Ratio	Notes on Teaching

Example

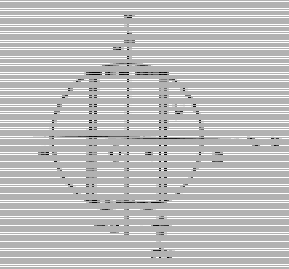
Find the volume of the sphere of radius a. Consider the sphere as a solid of revolution of a circle of radius a.
By the disc method:

$$\text{Volume of sphere} = \int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi(a^2 - x^2)dx = \frac{4\pi a^3}{3}$$



Alternatively the shell method may be used:

$$\text{Volume of sphere} = \int_0^a 2\pi x \cdot 2y \cdot dx = \int_0^a 4\pi x\sqrt{a^2 - x^2} \cdot dx = \frac{4\pi a^3}{3}$$



Due amount of examples should be provided. Volumes of cone, sphere, etc. are interesting examples.

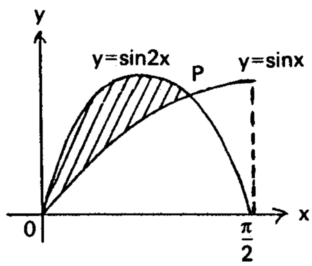
103

Detailed Content	Time Ratio	Notes on Teaching

Students should note that in solving this type of problems, finding points of intersection with the coordinate axes and those between the curves are often required, and hence, sketching the relevant curves beforehand is helpful.

Example

In the figure, the curves y = sin2x and y = sinx intersect at the point P. Find the coordinates of P and hence find the volume of solid of revolution of the shaded region about the x-axis.



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