## UNIT 3: INTERPOLATION AND LAGRANGE INTERPOLATING POLYNOMIALS

*Specific Objectives:*

1. To learn the concept of interpolation.
2. To learn Lagrange Interpolating Polynomial.
3. To apply Lagrange Interpolating Polynomial to approximate functions, and estimate the errors.

| Detailed Content | Time Ratio | Notes on Teaching |
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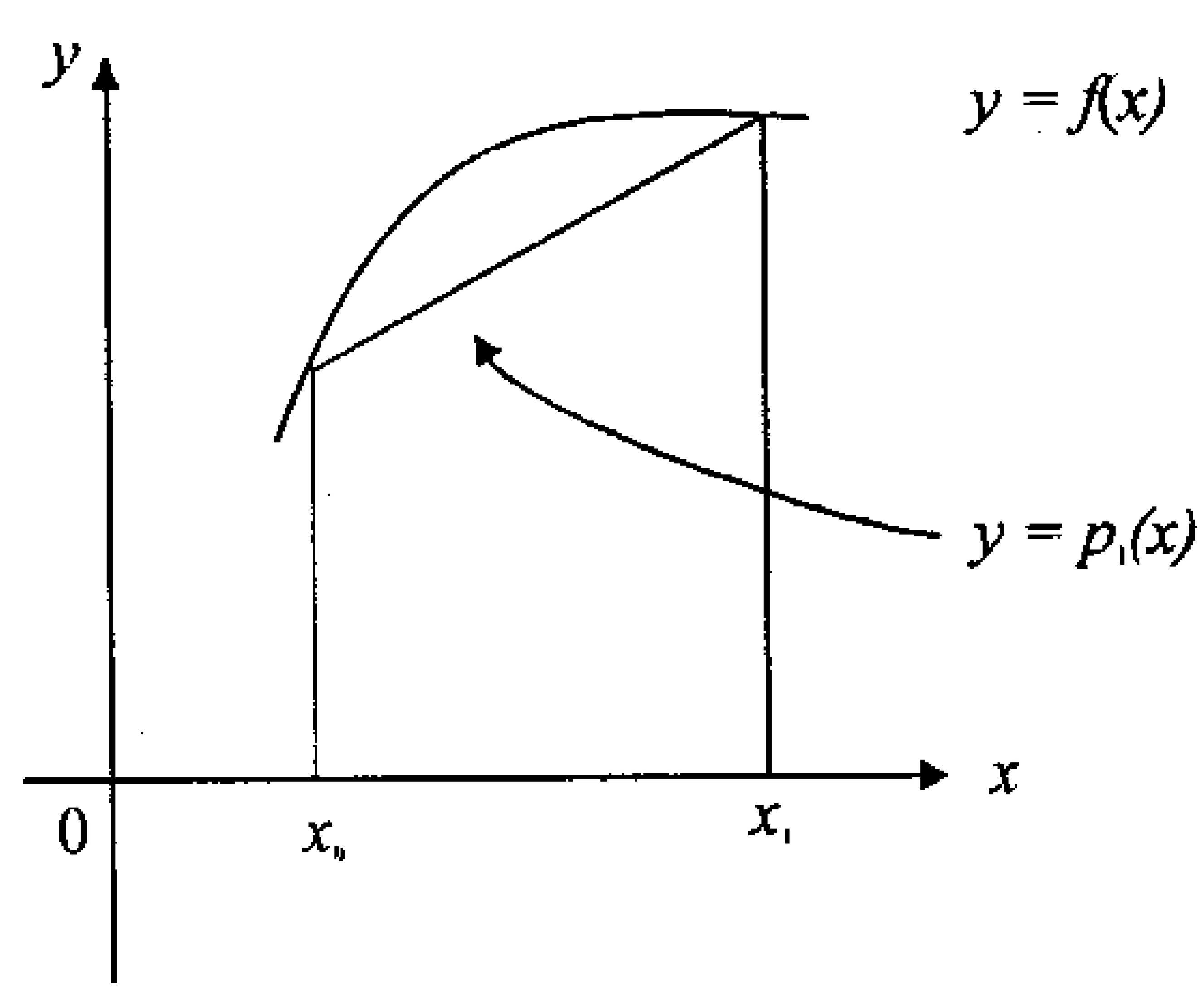
| **3.1 Interpolation and Interpolating Polynomials** | 1 | Students are expected to know the meaning of interpolation:<br><br>Interpolation involves estimating the values of a function $f(x)$ for arguments between $x_0, x_1, \ldots, x_n$ at which the values $f(x_0), f(x_1), \ldots, f(x_n)$ are known.<br><br>They are also expected to know that approximation by polynomial is one of the most heavily used in numerical methods. A polynomial $p(x)$ is used as a substitute for the function $f(x)$ because polynomials are easy to compute, only simple integral powers being involved; their derivatives and integrals are found without much difficulty and are themselves polynomials; roots of polynomial equations are also easy to locate. |
| **3.2 Construction of Lagrange Interpolating Polynomials** | 3 | As an introduction, teachers may demonstrate Lagrange Interpolating Polynomial (L.I.P.), $p_n(x)$ *for* $n = 1$. A graph as shown below may be used to give students a physical meaning.<br><br><br><br><br>Students should have no problem to see that they are just asked to approximate the curve $y = f(x)$ in between $x_0$ and $x_1$ by the polynomial $p_1(x) = a_1 x + a_0$. |

| Detailed Content | Time Ratio | Notes on Teaching |
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| | | Clearly, $p_1(x_0)=f(x_0), p_1(x_1)=f(x_1)$ and $p_1(x)=a_1x+a_0$.<br><br>We have<br><br>$p_1(x)-a_1x-a_0=0$<br>$f(x_0)-a_1x_0-a_0=0$<br>$f(x_1)-a_1x_1-a_0=0$<br><br>Eliminating $a_0$ and $a_1$, we get<br>$p_1(x)=f(x_0)\dfrac{x-x_1}{x_0-x_1}+f(x_1)\dfrac{x-x_0}{x_1-x_0}$. Similarly, teachers may proceed with the aid of a graph like that above to derive the second-degree L.I.P. and obtain<br><br>$p_2(x)=f(x_0)\dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}+f(x_1)\dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$<br>$+f(x_2)\dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$<br><br>In a similar manner the third-degree polynomial can also be deduced. For students' interest, the term 'Lagrange Multiplier Function' may be introduced.<br><br>Finally, teachers could help students to draw the conclusion that $p_n(x)$ $(n=1,\ 2,\ 3)$, is of degree $n$ and that $p_n(x_i)=f(x_i)$ at the $n$+1 tabulated points $x_i$. The extension of this fact to the general case is not a necessity. |
| **3.3 Use of Lagrange Interpolating Polynomial** | 2 | The use of *L.I.P.* should be demonstrated with examples.<br><br>*Example 1*<br><br>Give the four values of an unknown function at 0 ,1 , 2 , 4 as shown in the table. |

| $x_k$ | 0 | 1 | 2 | 4 |
| --- | --- | --- | --- | --- |
| $y_k$ | 1 | 1 | 2 | 5 |

The *L.I.P.* of degree 3 is $p_3(x)=\dfrac{1}{12}(-x^3+9x^2-8x+12)$. Students may also be required to evaluate some functional values for arguments lying between $x=0$ and $x=4$.

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*Example 2*

Using the data in the table

| | | | | | | |
|---|---|---|---|---|---|---|
| $v$ | 10 | 15 | 22.5 | 33.75 | 50.625 | 75.937 |
| $p$ | 0.300 | 0.675 | 1.519 | 3.417 | 7.689 | 17.300 |

Students may be required to apply *L.I.P.* to find the value of $p$ corresponding to $v = 21$ using various degrees of the *L.I.P.* chosen ($p = 1.350$ for $n = 1$ and $p = 1.323$ for $n = 2$). From this example, students should be able to realize that using the same number of different neighbouring points would yield different results.

For further illustration in class, common functions like sine and cosine functions are worth demonstrating. Intermediate functional values estimated using *L.I.P.* could be compared to actual values from a calculator. More interesting problems could be obtained by selecting a given table of data relating to an economic trend or the population of a country in a period and students are asked to estimate some missing data.

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| **3.4 Error Estimation of Interpolating Polynomial** | 3 | At this stage, teachers should remind students that *L.I.P.* is only a method of polynomial interpolation and that many other methods exist. For the abler students teachers may discuss with them the uniqueness of the interpolating polynomial. |

To begin error estimation, teachers may introduce the function $\pi(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)$, which may be used to express the coefficients of $f(x_i)$, i.e. $\dfrac{\pi(x)}{(x - x_i)\pi'(x_i)}$ in a given *L.I.P.*, in a more compact form, as

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\frac{\pi(x)}{(x - x_i)\pi'(x_i)} \quad ,n = 1,\ 2,\ 3.$$

By defining the error $e(x) = f(x) - p_n(x) = C\pi(x)$, where $C$ is a constant, and constructing the function $F(x) = f(x) - p_n(x) - C\pi(x)$ and choosing $x = \bar{x}$ ( where $x_0 \le \bar{x} \le x_n$ and $\bar{x} \ne x_i$, $i = 0, 1, 2, 3$ ), students could be led to apply Rolle's theorem repeatedly to arrive at $e(x) = \dfrac{f^{(n+1)}(\varepsilon)}{(n+1)!}\pi(x)$ where $\varepsilon$ lies in the tabulated interval.

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Students should then realize that if $f^{(n+1)}(\varepsilon)$ has an upperbound for $\varepsilon$ in the interval containing the tabulated points, say $M = \max\left|f^{(n+1)}(\varepsilon)\right|$ then the absolute error is given by $\left|e(x)\right| \le \left| M\dfrac{\pi(x)}{(n+1)!}\right|$.

Examples should be provided for illustration. The following is one such example.

*Example*

Given the function $f(x) = \sin\dfrac{\pi x}{2}$ takes the following values in the table

| $x_k$ | 0 | 1 | 2 |
|---|---|---|---|
| $y_k$ | 0 | 1 | 0 |

students may be required to show that $\left|e(x)\right| \le \left|\dfrac{\pi^3}{8}x(x-1)(x-2)\right|$ and then to compute this estimate at $x = 0.5$ and compare it with the actual error.

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