Chapter 5 Exemplars

Exemplar 1 **A. M. $\geq$ G. M.**

Exemplar 2 **Plane Area**

Exemplar 3 **The Binomial Theorem**

Exemplar 1:

A. M. ≥ G. M.

Objective: To prove A. M. ≥ G. M. without the application of Backward Induction

Pre-requisite knowledge:
(1) The Principle of Mathematical Induction
(2) Fundamental techniques in proving absolute inequalities

Description of the Activity:

Let $A_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and $G_n = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$, where $a_1, a_2, \ldots, a_n$ are n positive numbers. It was suggested in the *Syllabuses for Secondary Schools – Pure Mathematics (Advanced Level) 1992* that teachers may prove $A_n \geq G_n$ by backward induction if required. However, backward induction is deleted from this Curriculum. Some suggestions to prove the inequality are as follows:

<u>Method 1</u>

It is obvious that $A_1 = G_1$ and $A_2 \geq G_2$.

Assume that $A_k \geq G_k$ is true, where k is a positive integer greater than or equal to 2.

When $n = k + 1$,

Case (i) If $a_1 = a_2 = \cdots = a_{k+1}$, then $A_{k+1} = G_{k+1}$.

Case (ii) If not all $a_1, a_2, \ldots, a_{k+1}$ are equal, we may assume, without loss of generality, that $a_1 \leq a_2 \leq \cdots \leq a_{k+1}$ *and* $a_1 < a_{k+1}$.

It follows that $\dfrac{a_1}{G_{k+1}} < 1$, $\dfrac{a_{k+1}}{G_{k+1}} > 1$.

Let $y = a_1 a_{k+1}$. Since $A_k \geq G_k$, we have

$$\frac{y}{(G_{k+1})^2} + \frac{a_2}{G_{k+1}} + \cdots + \frac{a_k}{G_{k+1}} \geq k \sqrt[k]{\frac{y}{(G_{k+1})^2} \cdot \frac{a_2}{G_{k+1}} \cdot \cdots \cdot \frac{a_k}{G_{k+1}}} = k \sqrt[k]{\frac{a_1 a_{k+1} a_2 \ldots a_k}{(G_{k+1})^{k+1}}} = k$$

Adding $\dfrac{a_1}{G_{k+1}} + \dfrac{a_{k+1}}{G_{k+1}} - \dfrac{y}{(G_{k+1})^2}$ to both sides of the inequality, we have

$$\frac{a_1}{G_{k+1}} + \frac{a_2}{G_{k+1}} + \cdots + \frac{a_k}{G_{k+1}} + \frac{a_{k+1}}{G_{k+1}} \geq k + \frac{a_1}{G_{k+1}} + \frac{a_{k+1}}{G_{k+1}} - \frac{y}{(G_{k+1})^2}$$

$$= k + 1 + \frac{a_1}{G_{k+1}} - 1 + \frac{a_{k+1}}{G_{k+1}} - \frac{a_1 a_{k+1}}{G_{k+1}} \cdot \frac{1}{G_{k+1}}$$

$$= k+1+\left(1-\frac{a_1}{G_{k+1}}\right)\left(\frac{a_{k+1}}{G_{k+1}}-1\right)$$

$\because \quad \frac{a_1}{G_{k+1}}<1, \quad \frac{a_{k+1}}{G_{k+1}}>1, \quad \therefore \quad \left(1-\frac{a_1}{G_{k+1}}\right)\left(\frac{a_{k+1}}{G_{k+1}}-1\right)>0.$

Thus, we have $\frac{a_1}{G_{k+1}}+\frac{a_2}{G_{k+1}}+\cdots+\frac{a_k}{G_{k+1}}+\frac{a_{k+1}}{G_{k+1}}>k+1.$

$\therefore \quad A_{k+1}>G_{k+1}$ holds.

From cases (i) and (ii), we have $A_{k+1}\ge G_{k+1}$

By the principle of mathematical induction, $A_n \ge G_n$ is true for all natural numbers n.

<u>Method 2</u>

It is obvious that $A_1=G_1$ and $A_2\ge G_2$.

Assume that $A_k\ge G_k$ is true, where k is a positive integer greater than or equal to 2.

Let the geometric mean and the arithmetic mean of $\underbrace{A_{k+1}, A_{k+1}, \ldots, A_{k+1}}_{(k-1)\ terms}$ and a_{k+1} be M and L respectively. Then $M=(a_{k+1}A_{k+1}{}^{k-1})^{\frac{1}{k}}$ and $L=\frac{1}{k}\left[a_{k+1}+(k-1)A_{k+1}\right]$.

By the induction hypothesis, $M\le L$ and

$$\left(G_{k+1}{}^{k+1}A_{k+1}{}^{k-1}\right)^{\frac{1}{2k}}=\left(a_1a_2\cdots a_k a_{k+1}A_{k+1}{}^{k-1}\right)^{\frac{1}{2k}}=\left[(a_1a_2\cdots a_k)^{\frac{1}{k}}\left(a_{k+1}A_{k+1}{}^{k-1}\right)^{\frac{1}{k}}\right]^{\frac{1}{2}}$$

$$=(G_kM)^{\frac{1}{2}}$$

$$\le\frac{1}{2}(G_k+M)$$

$$\le\frac{1}{2}(A_k+L) \qquad \text{since } G_k\le A_k \text{ and } M\le L$$

$$=\frac{1}{2}\{A_k+\frac{1}{k}\left[a_{k+1}+(k-1)A_{k+1}\right]\}$$

$$=\frac{1}{2k}\{a_{k+1}+kA_k+(k-1)A_{k+1}\}.$$

$$=\frac{1}{2k}\{(k+1)A_{k+1}+(k-1)A_{k+1}\}$$

$$=A_{k+1}$$

i.e. $(G_{k+1})^{k+1}(A_{k+1})^{k-1}\le(A_{k+1})^{2k}$

$\therefore G_{k+1}\le A_{k+1}$ holds.

By the principle of mathematical induction, $A_n\ge G_n$ is true for all natural numbers n.

Method 3

It is obvious that $A_1 = G_1$ and $A_2 \ge G_2$.

Assume that $A_k \ge G_k$ is true, where k is a positive integer greater than or equal to 2.

When $n = k+1$,

Case (i) If $a_1 = a_2 = \cdots = a_{k+1}$, then $A_{k+1} = G_{k+1}$.

Case (ii) If not all $a_1, a_2, \ldots, a_{k+1}$ are equal, we may assume, without loss of generality, that $a_1 \le a_2 \le \cdots \le a_{k+1}$ *and* $a_1 < a_{k+1}$.

Since $A_k \ge G_k$, we have

$$a_1 + a_2 + \cdots + a_k + a_{k+1} \ge k\sqrt[k]{a_1 a_2 \cdots a_k} + a_{k+1}$$

$$= \left(k\sqrt[k]{\frac{a_1 a_2 \cdots a_{k+1}}{(a_{k+1})^{k+1}}} + 1 \right) a_{k+1}$$

$$= (kr^{k+1} + 1)\, a_{k+1}$$

where $r^{k(k+1)} = \dfrac{a_1 a_2 \ldots a_{k+1}}{(a_{k+1})^{k+1}}$ and $r^{k(k+1)} < 1$ with $r > 0$

Since $0 < r^{k(k+1)} < 1$, then $0 < r < 1$ and $r^k < r^{k-1} < r^{k-2} < \ldots\ldots < r^3 < r^2 < r$.

As $\dfrac{1-r^{k+1}}{1-r} = 1 + r + \ldots + r^k > \underbrace{r^k + r^k + \ldots + r^k}_{(k+1)\, terms} = (k+1)r^k$,

$\therefore\ 1 - r^{k+1} > (k+1)r^k(1-r)$.

Hence, we have $1 - r^{k+1} + (k+1)r^{k+1} > (k+1)r^k$.

$\therefore\ kr^{k+1} + 1 > (k+1)r^k$

Since $a_1 + a_2 + \cdots + a_k + a_{k+1} \ge (kr^{k+1} + 1)\, a_{k+1}$

$\therefore\ a_1 + a_2 + \cdots + a_k + a_{k+1} > (k+1)r^k a_{k+1}$

$$= (k+1)\sqrt[k+1]{\frac{a_1 a_2 \cdots a_{k+1}}{(a_{k+1})^{k+1}}}\ a_{k+1}$$

$$= (k+1)\sqrt[k+1]{a_1 a_2 \cdots a_k a_{k+1}}$$

$\therefore\ A_{k+1} > G_{k+1}$ holds.

From cases (i) and (ii), we have $A_{k+1} \ge G_{k+1}$.

By the principle of mathematical induction, $A_n \ge G_n$ is true for all natural numbers n.

Method 4

It is obvious that $A_1 = G_1$ and $A_2 \geq G_2$.

Assume that $A_k \geq G_k$ is true, where k is a positive integer greater than or equal to 2.

When $n = k+1$,

Case (i) If $a_1 = a_2 = \cdots = a_{k+1}$, then $A_{k+1} = G_{k+1}$.

Case (ii) If not all $a_1, a_2, \ldots, a_{k+1}$ are equal, we may assume, without loss of generality, that $a_1 \leq a_2 \leq \cdots \leq a_{k+1}$ *and* $a_1 < a_{k+1}$.

It follows that $a_{k+1} > \sqrt[k]{a_1 a_2 \cdots a_k} = G_k$, and so $a_{k+1} - G_k > 0$.

By the induction hypothesis,

$$
\begin{aligned}
A_{k+1} &= \frac{1}{k+1}(a_1 + a_2 + \cdots + a_k + a_{k+1}) \\
&= \frac{kA_k + a_{k+1}}{k+1} \\
&\geq \frac{kG_k + a_{k+1}}{k+1} \\
&= G_k + \frac{a_{k+1} - G_k}{k+1}
\end{aligned}
$$

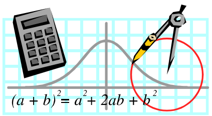
By the Binomial Theorem, we have

$$
\begin{aligned}
(A_{k+1})^{k+1} &\geq \left(G_k + \frac{a_{k+1} - G_k}{k+1} \right)^{k+1} \\
&= (G_k)^{k+1} + (k+1)(G_k)^k \left(\frac{a_{k+1} - G_k}{k+1}\right) + \cdots \qquad \text{(all terms are positive)} \\
&> (G_k)^{k+1} + (G_k)^k (a_{k+1} - G_k) \\
&= (G_k)^k a_{k+1} \\
&= (G_{k+1})^{k+1}
\end{aligned}
$$

$\therefore A_{k+1} > G_{k+1}$ holds.

From cases (i) and (ii), we have $A_{k+1} \geq G_{k+1}$.

By the principle of mathematical induction, $A_n \geq G_n$ is true for all natural numbers n.



Exemplar 2:

Plane Area

Objective: To prove that the area bounded by the curve with parametric equations x=$x(t)$, y=$y(t)$, and the lines OA, OB is $\frac{1}{2}\int_{t_0}^{t_1}(x\frac{\mathrm{d}y}{\mathrm{d}t}-y\frac{\mathrm{d}x}{\mathrm{d}t})\mathrm{d}t$ ……………(*)

where the parameters of A and B are t_0 and t_1 respectively.

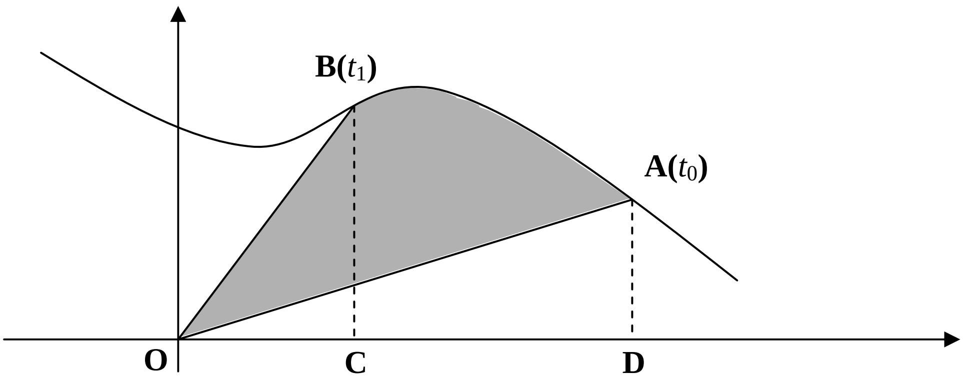
Pre-requisite knowledge:

(1) The application of definite integrals to find the area under a curve in Cartesian form.

(2) The Fundamental Theorem of Integral Calculus.

Description of the Activity:

Many teachers used to prove the formula (*) by means of formulae related to the polar coordinate system. The formula (*) can be readily derived from $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$, which gives the area bounded by the curve with the polar equation $r = f(\theta)$ and the two radii with radius vectors $\theta = \alpha$ and $\theta = \beta$. The contents related to the polar coordinate system are deleted from this curriculum. A suggestion to prove the formula (*) is as follows:



A curve with the parametric equations x=$x(t)$, y=$y(t)$ is shown in the diagram above. The parameters of A and B are t_0 and t_1 respectively. Without loss of generality, we may assume that, when the parameter t increases, the curve is continuous and goes in the anticlockwise direction.

Area of the shaded region = Area of Δ BOC + Area of ABCD – Area of Δ AOD

$$= \frac{x(t_1)\ y(t_1)}{2} + \int_{t=t_1}^{t=t_0} y\,\mathrm{d}x - \frac{x(t_0)\ y(t_0)}{2}$$

$$= \frac{x(t_1)\ y(t_1)}{2} + \int_{t_1}^{t_0} y(t)x'(t)\,\mathrm{d}t - \frac{x(t_0)\ y(t_0)}{2}$$

$$= \frac{1}{2}[\ x(t_1)\ y(t_1) - x(t_0)\ y(t_0)] - \int_{t_0}^{t_1} y(t)x'(t)\,\mathrm{d}t$$

By the Second fundamental Theorem of Integral Calculus (p.69 in Appendix 2),

we have $x(t_1)\ y(t_1) - x(t_0)\ y(t_0) = \int_{t=t_0}^{t=t_1} \mathrm{d}[x(t)\,y(t)]$.

Hence, we have

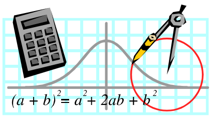
Area of the shaded region $= \frac{1}{2}\int_{t=t_0}^{t=t_1} \mathrm{d}\,[x(t)\,y(t)] - \int_{t_0}^{t_1} y(t)x'(t)\,\mathrm{d}t$

$$= \frac{1}{2}\int_{t_0}^{t_1} [x(t)\,y'(t) + x'(t)\,y(t)]\,\mathrm{d}t - \int_{t_0}^{t_1} y(t)x'(t)\,\mathrm{d}t$$

$$= \frac{1}{2}\int_{t_0}^{t_1} [x(t)\,y'(t) - x'(t)\,y(t)]\,\mathrm{d}t$$

$$= \frac{1}{2}\int_{t_0}^{t_1} [x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}]\,\mathrm{d}t$$

i.e. The area bounded by the curve with parametric equations x=$x(t)$, y=$y(t)$ and the lines OA, OB is $\frac{1}{2}\int_{t_0}^{t_1} [x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}]\,\mathrm{d}t$.



Exemplar 3:

The Binomial Theorem

Objective: To prove the Binomial Theorem for positive integral indices.

Pre-requisite knowledge: The relations between the roots and coefficients of a polynomial equation with real coefficients.

Description of the Activity:

Most teachers apply the Principle of Mathematical Induction to prove the Binomial Theorem for positive integral indices. An alternative way to prove that, for positive integers n,

$$(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + ...+ C_r^n a^{n-r}b^r + ...+ C_{n-1}^n ab^{n-1} + C_n^n b^n$$

is as follows:

Let $(x+b)^n = a_n x^n + a_{n-1}x^{n-1} + ... + a_k x^k + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_{n-1}, a_n$ are real constants.

Since the equation $(x+b)^n = 0$ has n repeated roots $x = -b$,

the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_k x^k + ... + a_1 x + a_0 = 0$ has n roots $x_1, x_2, ..., x_{n-1}, x_n$

with $x_1 = x_2 = ... = x_{n-1} = x_n = -b$.

By using the relations between the roots and coefficients of a polynomial equation with real coefficients,

$$\begin{cases} x_1 + x_2 + ...+ x_n = -\dfrac{a_{n-1}}{a_n}, \\ x_1x_2 + x_1x_3 + ...+ x_{n-1}x_n = \dfrac{a_{n-2}}{a_n}, \\ x_1x_2x_3 + x_1x_2x_4 + ...+ x_{n-2}x_{n-1}x_n = -\dfrac{a_{n-3}}{a_n}, \\ \dots \quad \dots \quad \dots \\ x_1x_2...x_k + x_1x_2...x_{k+1} + ...+ x_{n-k+1}x_{n-k+2}...x_n = (-1)^k \dfrac{a_{n-k}}{a_n}, \\ \dots \quad \dots \quad \dots \\ x_1x_2...x_n = (-1)^n \dfrac{a_0}{a_n}. \end{cases}$$

In the k^{th} equality above, the left-hand side is the sum of the product of k terms of x_i .

Since $x_1 = x_2 = ... = x_{n-1} = x_n = -b$,

$$\therefore \quad (-b)^k C_k^n = (-1)^k \frac{a_{n-k}}{a_n} .$$

As a_n is the coefficient of x_n in the expansion of $(x+b)^n$, it is obvious that $a_n=1$,

$$\therefore \quad a_{n-k} = C_k^n b^k \quad (k =1, 2, \ldots\ldots, n).$$

i.e. $a_n = 1 = C_0^n$, $a_{n-1} = C_1^n b$, ……, $a_{n-k} = C_k^n b^k$, ……, $a_0 = C_n^n b^n$

$$\therefore \quad (x+b)^n = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_k x^k + ... + a_1 x + a_0$$

$$= C_0^n x^n + C_1^n b x^{n-1} + C_2^n b^2 x^{n-2} + ... + C_{n-k}^n b^{n-k} x^k + ... + C_{n-1}^n b^{n-1} x + C_n^n b^n .$$

Putting $x=a$, we have

$$(a+b)^n = C_0^n a^n + C_1^n b a^{n-1} + C_2^n b^2 a^{n-2} + ... + C_{n-k}^n b^{n-k} a^k + ... + C_{n-1}^n b^{n-1} a + C_n^n b^n$$

$$= C_0^n a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + ... + C_{n-k}^n a^k b^{n-k} + ... + C_{n-1}^n a b^{n-1} + C_n^n b^n .$$