Unit A2: Functions

Objective : (1) To recognize function as a fundamental tool in other branches of mathematics.

(2) To sketch and to describe the shapes of different functions.

Detailed Content	Time Ratio	Notes on Teaching
2.1 Functions and their graphs	2	Students should be taught with a clear definition of function, however rigorous treatment is not expected. The following version may be adopted: $f: A \to B$, f is a function from A to B if every element in A associated with an unique element in B. A is called the domain of f, B the range of f. For an element x in A, the element in B which is associated with x under f usually denoted by f(x) is called the image of x under f and $f[A]$ denotes the set of images of A under f. Special emphasis should be put on real-valued functions since they are most useful in the discussion of other mathematical topics in this syllabus. Students are also expected to be able to plot/ sketch the graph of functions.
2.2 Properties and operations of functions	4	Clear definitions of injective, surjective and bijective functions should be taught so that students are able to distinguish them and to apply the knowledge to solve problems concerned. The following suggested versions may be adopted: A function $f: A \to B$ is (i) injective (one-to-one) if and only if for elements a_1, a_2 in A, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$, or equivalently, $f(a_1) = f(a_2)$ implies $a_1 = a_2$; (ii) surjective (onto) if and only if $f[A] = B$ i.e. every element in B is the image of an element in A; (iii) bijective (one-to-one correspondence) if and only if f is injective and surjective. At this juncture, teachers may provide sufficient preparation on the part of the students so that the concept of inverse function denoted by f^{-1} can be easily figured out and the property that f is bijective if and only if its inverse function f^{-1} exists. Moreover the property that the graphs of a function and its inverse (if exists) are reflections about the line y = x should be studied with adequate illustrations. Students should be able to distinguish odd, even, periodic, increasing and decreasing 'functions. In this connection, clear definitions should be provided. For increasing and decreasing functions, prior to the teaching of differential calculus, teachers may adopt the following suggested .definitions: (i) f(x) is increasing (strictly increasing) if and only if $x_2 > x_1$ implies $f(x_2) \geq f(x_1)$ $(f(x_2) > f(x_1))$;

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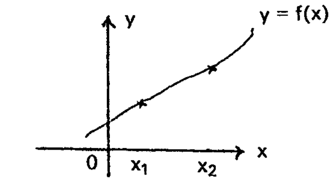
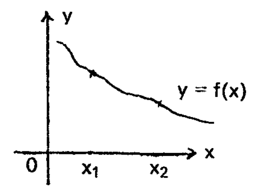
		(ii) f(x) is decreasing (strictly decreasing) if and only if $x_2 > x_1$ implies $f(x_2) \leq f(x_1)$ $(f(x_2) < f(x_1))$.



f(x) is strictly increasing

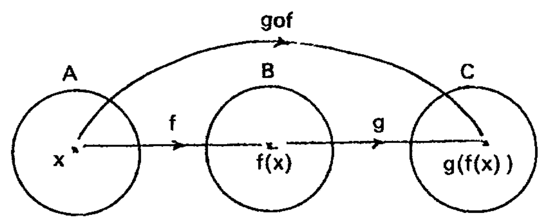


f(x) is strictly decreasing

These properties may be useful in sketching curves and in evaluating definite integrals, etc.

Concerning the operations with functions, teachers should discuss with the students that, for functions f and g, f + g, f – g, f × g and $\frac{f}{g}$ (provided $g(x) \neq 0$ for all values of x concerned) are again functions. However regarding the composition of functions which is very useful especially in teaching the chain rule in differential calculus, it is desirable to give ample simple examples so as to support students' mastery of the concepts. Teachers may consider the following suggestion with due emphasis directed to real-valued functions:

If $f: A \to B$ and $g: B \to C$ are functions, then the composition of f and g is the function $g \circ f: A \to C$ such that $g \circ f(x) = g(f(x))$ for all element x in A.



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2.3 Algebraic functions	2	Students should be able to recognize the- following algebraic functions: *(a)* polynomial functions *(b)* rational functions *(c)* power functions x^{α} where α is rational *(d)* other algebraic functions derived from the above-mentioned ones through addition, subtraction, multiplication, division and composition like $\sqrt{x^2+1}$
2.4 Trigonometric functions and their formulae	14	Students should be able to sketch the graphs of the six trigonometric functions and their inverse functions. The basic relations like $\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ should also be included in the discussion with the students. Simplification of these functions at $(\frac{n\pi}{2} \pm \theta)$ for odd and even n and proving of identities are expected. The knowledge and related applications of the following are also expected: (1) compound angle formulae $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (2) Multiple angle formulae $\sin 2A = 2\sin A \cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $= 2\cos^2 A - 1$ $= 1 - 2\sin^2 A$ $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ $\sin 3A = 3\sin A - 4\sin^3 A$ $\cos 3A = 4\cos^3 A - 3\cos A$ $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

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		(3) Half-angle formulae $\sin^2\frac{A}{2} = \frac{1}{2}(1-\cos A)$ $\cos^2\frac{A}{2} = \frac{1}{2}(1+\cos A)$ $\sin A = \frac{2t}{1+t^2}$, $\cos A = \frac{1-t^2}{1+t^2}$ with $t \equiv \tan\frac{A}{2}$ (4) sum and product formulae $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ $\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$ $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$ $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ As a matter of fact, most of the formulae are direct consequence from the basic ones thus teachers should encourage their students to work out the proofs for

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		themselves. The transformation of the expression acosx + bsinx into the form rsin(x + α) or rcos(x + β) should be discussed. Applications of the above formulae in proving identities and in solving trigonometric equation should be included.
2.5 Exponential and logarithmic functions	6	Students should know the relation between the exponential and logarithmic functions, viz. one is the inverse function of the other. The common definition of logarithm, like $\log_a x = y$ if and only if $x = a^y$ with a > 0 and a $\neq$ 1 should be revised. As for logarithmic functions of a variable x, students should know that they are functions of logx or of the logarithm of some function of x, like $(\log_{10} x)^2$ and $\log_e(1+\tan x)$ etc. Some common properties of the logarithmic function should be studied: For $f(x) = \log_a x$ with a > o and a $\neq$ 1; (i) $f(x)$ is defined for x > 0 only; (ii) $f(x)$ is an increasing function if a > 1 and is a decreasing function if 0 < a < 1; (iii) for b, c > 0 and b $\neq$ 1, $\log_a c = \dfrac{\log_b c}{\log_b a}$; (iv) $f(a) = \log_a a = 1$; (v) $f(1) = \log_a 1 = 0$. For the exponential function, a parallel treatment should be provided, viz, a function of the form a^x where a is a positive constant and x a variable is called an exponential function. Those common properties of the exponential function include the following: For $f(x) = a^x$ with a > 0 and a $\neq$ 1, (i) f(x) is defined for all real x; (ii) (x) is an increasing function if a > 1 and is a decreasing one if 0 < a < 1; (iii) $f(0) = a^0 = 1$. Students should be able to sketch the graph for (i) the logarithmic function $f(x) = \log_a x$ for the cases a > 1 and 0 < a < 1 (ii) the exponential function also for the cases a > 1 and 0 < a < 1.

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		At this juncture, teachers may embark on the following important results so as to extend students' perspective on logarithmic and exponential functions (i) $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ (ii) $\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$ and (iii) $\log_e x = \int_1^x \frac{1}{t}\,dt$. (Please note that the third one is optional which could be taken as an alternative definition for $\log_e x$ or written as $\ell n\, x$). The general functional properties of logarithmic and exponential functions should also be mentioned. Logarithmic function $f(x) = \log_a x$ with a > 0 , a $\neq$1. (i) $f(x) + f(y) = f(xy)$ (ii) $f(x) - f(y) = f(\frac{x}{y})$ (iii) $f(x^n) = nf(x)$ and Exponential function $g(x) = a^x$ with a > 0 (i) $g(x+y) = g(x)\cdot g(y)$ (ii) $g(x-y) = \dfrac{g(x)}{g(y)}$ (iii) $g(nx) = (g(x))^n$ And, in particular, the importance of the functions e^x and $\ell n\, x$ in the study of mathematics should be emphasized.
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