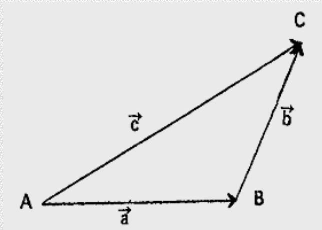
Unit A7: Vectors in R^2 and R^3

Objective: (1) To study the operations of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.
(2) To understand the concept of linearly dependent vectors and linearly independent vectors.
(3) To apply vectors in geometrical problems.

Detailed Content	Time Ratio	Notes on Teaching
7.1 Definition of Vectors and scalars	1	To begin with this unit, the difference between vectors and scalars should be explained to students. The representation of a vector, both pictorial and written, should be introduced. The current notations of vectors (such as $\overrightarrow{AB}$, **AB**, $\vec{a}$, **a**) and their magnitudes (such as $\lvert\overrightarrow{AB}\rvert$, |**AB**|, $\lvert\vec{a}\rvert$, |**a**|) are to be taught. The terms null vector, unit vector, equal vectors, negative vector, collinear vectors and coplanar vectors should also be defined.
7.2 Operations of vectors	7	Student should know the laws of vector addition (namely, the triangle law, the parallelogram law, and the polygon law), the subtraction of vectors and the multiplication of a vector by a scalar.

(i) Triangle law



$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
or
$\vec{a}+\vec{b}=\vec{c}$

It should be pointed out to the student that, when using the law to find $\vec{a}+\vec{b}$, the end point of vector $\vec{a}$ must coincide with the initial point of vector $\vec{b}$. It should be noted that the validity of the law still holds when A, B, C are collinear points.

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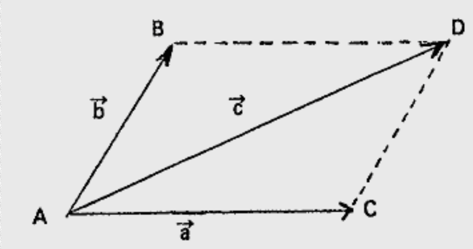
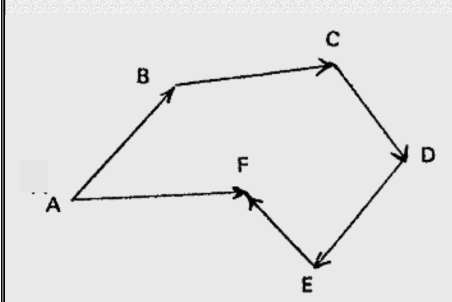
(ii) Parallelogram law



$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$
or
$\vec{a}+\vec{b}=\vec{c}$

In a similar manner, teachers should remind the students that the initial points of vectors $\vec{a}$ and $\vec{b}$ must be coincident and in either of the above cases, $\vec{c}$ can also be regarded as the resultant of $\vec{a}$ and $\vec{b}$. The equivalence of the triangle law and the parallelogram law is worth discussing.

(iii) Polygon law



$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{AF}$

The laws of the vector algebra like commutative law, associative law and distributive law should also be made known to students. The following diagrams may be useful in illustrating these properties.

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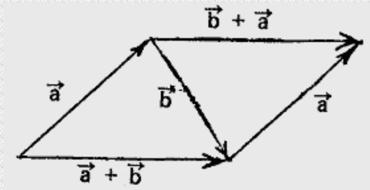
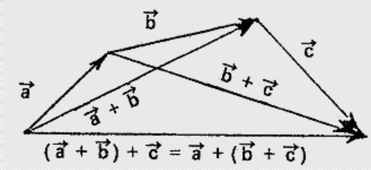
(a) Commutative law of addition: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$



(b) Associative law of addition: $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$



(c) Associative law for scalar multiplication:

$(\alpha\beta)\vec{a}=\alpha\,(\beta\,\vec{a})$

Distributive laws for scalar multiplication:

$\alpha\,(\vec{a}+\vec{b})\;=\alpha\,\vec{a}+\alpha\,\vec{b}$

$(\alpha+\beta)\vec{a}=\alpha\,\vec{a}+\beta\,\vec{a}$

After understanding the concept of scalar multiplication, students should have no difficulty to deduce the result that

if $\vec{a}$, $\vec{b}$ are non-zero vectors such that $\vec{a}=\alpha\,\vec{b}$ for some scalar α, then $\vec{a}\,//\,\vec{b}$.

It should be made clear to students concerning the resolution of a vector into component vectors, and the specification of a vectors as a sum of component vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. The resolution of vectors in $\mathbf{R}^2$ can be introduced with the following examples. In the first example, $\vec{r}$ is resolved into two components $5\vec{a}$ and $4\vec{b}$ in the directions of $\vec{a}$ and $\vec{b}$ respectively. This can be generalized to $\vec{r}=\alpha\,\vec{a}+\beta\,\vec{b}$ where $\vec{a}$ and $\vec{b}$ are non-collinear vectors in $\mathbf{R}^2$ and $\vec{r}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors in $\mathbf{R}^3$, for scalars α, β and γ.

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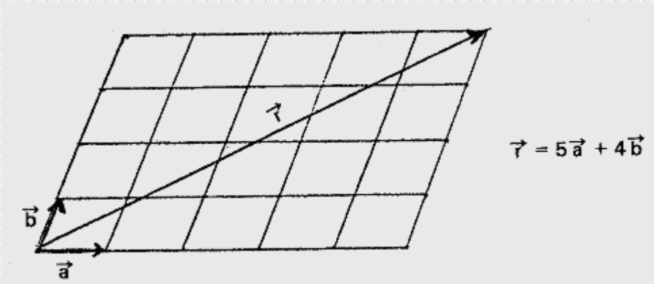
Examples:

1.



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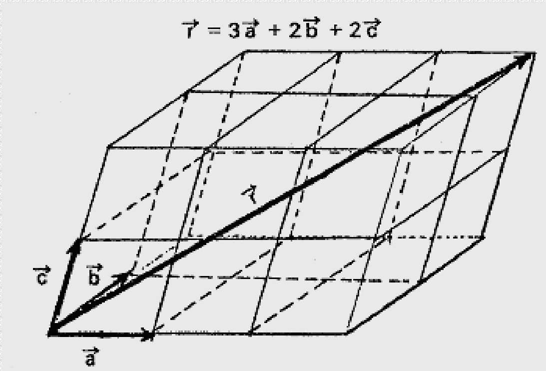

Furthermore, scalar multiplication, addition and subtraction of vectors in terms of component vectors should be discussed.

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<tr><td>7.3 Resolution of vectors in the rectangular coordinate system</td><td>2</td><td>

The face that $\vec{i}$, $\vec{j}$ and $\vec{k}$ represent the unit vectors in the directions of the positive x-, y- and z-axis respectively and that any vector in $\mathbf{R}^2$ and $\mathbf{R}^3$ can be expressed in the form $a\vec{i}+b\vec{j}+c\vec{k}$ should be explained in detail.

Students are required to be familiar with the following properties of vectors in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

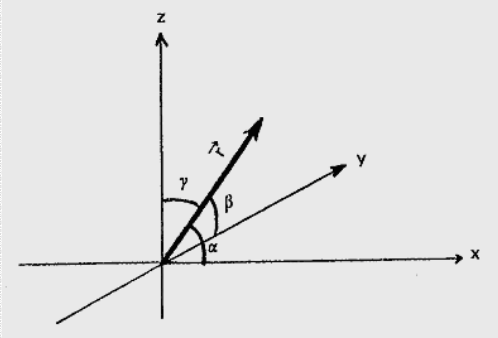
(i) $\left|a\vec{i}+b\vec{j}+c\vec{k}\right|=\sqrt{a^2+b^2+c^2}$;

(ii) two vectors $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$ and $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$ are parallel if $a_1:b_1:c_1=a_2:b_2:c_2$.

Moreover the meaning of direction ratio, direction cosines and direction angle of $\vec{r}$ should be explained with the help of diagrams, and the following properties should be discussed.

(i) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

(ii) $\dfrac{\vec{r}}{|\vec{r}|}=\cos\alpha\,\vec{i}+\cos\beta\,\vec{j}+\cos\gamma\,\vec{k}$



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<tr><td>7.4 Linear combination of vectors</td><td>4</td><td>

The following definitions should be taught:

(i) Let $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ be a set of vectors. An expression of the form $\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}$, where λ_1 , λ_2 , λ_3 , …, λ_n are scalars, is called a linear combination of the vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$. If the scalars λ's are not all zero, it is called a non-trivial linear combination, otherwise it is a trivial linear combination.

(ii) A set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is said to be linearly dependent if there exists a non-trivial linear combination of them equal to the zero vector. i.e.
$\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}=\vec{0}$ where
$\lambda_i\neq 0$ for some $i=1,2,3,\ldots,n$.

(iii) A set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is said to be linearly independent if the only linear combination of them equal to zero is the trivial one. i.e.
if $\lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}+\ldots+\lambda_n\vec{r_n}=\vec{0}$ then
$\lambda_1=\lambda_2=\lambda_3=\ldots=\lambda_n=0$.

Students should be helped to deduce an immediate result from (ii) that the set of vectors $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$, …, $\vec{r_n}$ is linearly dependent if and only if one of the vectors is a linear combination of the others in the set.

The geometrical significance of linear dependence of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ like the following should be elaborated.

(i) vectors $\vec{r_1}$ and $\vec{r_2}$ of $\mathbf{R}^2$ are linearly dependent if and only if they are parallel;

(ii) vectors $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ of $\mathbf{R}^3$ are linearly dependent if and only if they are coplanar.

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<tr><td>7.5 Scalar (dot) product and vector (cross) product</td><td>6</td><td>

The definition of the scalar product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\cdot\vec{b}$, in its usual context that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$, should be taught and the following properties discussed.

1. commutative law of scalar product: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
2. distributive law of scalar product: $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
3. $\vec{a}\cdot\vec{a}=|\vec{a}|^2$
4. two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b}=0$
5. $\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

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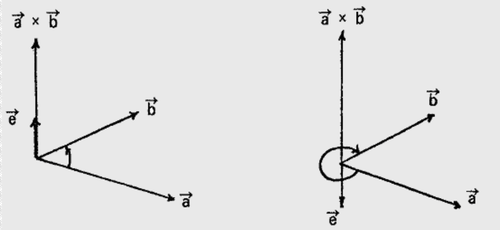
In particular, teachers should point out that, with references to the fifth property listed above, the scalar product can be used to find the angle between two vectors expressed in Cartesian components. In this connection students may be hinted to show that $\vec{i}\cdot\vec{i}=1$, $\vec{i}\cdot\vec{j}=0$ etc and the result that for $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$, $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$, $\vec{r_1}\cdot\vec{r_2}=a_1a_2+b_1b_2+c_1c_2$.

As for vectors product, the definition must be clearly provided. Special attention should be directed to the proper orientation of the right-hand system.

The vector product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\times\vec{b}$ is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\vec{e}$, where

(i) $\vec{e}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$;

(ii) θ is the angle from $\vec{a}$ to $\vec{b}$ measured in the direction determined by $\vec{e}$ according to the right-hand rule.



Discussion on the following properties is essential.

(i) $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$

(ii) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ and

$(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$

(iii) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

(iv) $|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$

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7.6 Application of vectors in geometry	11	

Students may be required to work out for themselves the results like $\vec{i}\times\vec{i}=\vec{0}$, $\vec{i}\times\vec{j}=\vec{k}$ etc as a prelude to deduce the result that for vectors expressed in Cartesian components $\vec{r_1}=a_1\vec{i}+b_1\vec{j}+c_1\vec{k}$ and $\vec{r_2}=a_2\vec{i}+b_2\vec{j}+c_2\vec{k}$

$\vec{r_1}\times\vec{r_2}=(b_1c_2-b_2c_1)\vec{i}+(c_1a_2-c_2a_1)\vec{j}+(a_1b_2-a_2b_1)\vec{k}$ or

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

Moreover, the properties that

(i) two non-zero vectors $\vec{a}$, $\vec{b}$ are parallel if and only if $\vec{a}\times\vec{b}=\vec{0}$

(ii) $|\vec{a}\times\vec{b}|$ may be interpreted as the area of the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$.

are helpful in reinforcing students' mastery of the concept.

Teachers are advised to provide students with detailed explanation and adequate discussion as well as exemplification on the use of relative vectors including position vector and displacement vector. The usual convention that the position vectors of points P and Q with respect to a reference point O are denoted by

$\overrightarrow{OP}$, $\overrightarrow{OQ}$ or $\vec{p}$, $\vec{q}$ respectively

and that

$\overrightarrow{PQ}=\vec{q}-\vec{p}$ should be highlighted.

The following results should be derived whilst other related generalization are also worth discussing for consolidation.

Position vector of point of division:

Let $\vec{a}$, $\vec{b}$ and $\vec{p}$ be the respective position vectors of A, B and P with reference to the point O. If P divides the line segment AB in the ratio of m:n then

$$\vec{p}=\frac{n\vec{a}+m\vec{b}}{m+n}$$

The different treatments for points of internal and external division should also be discussed. Sometimes the form $\vec{p}=\dfrac{k\vec{a}+\vec{b}}{1+k}$ should be preferred because, with adequate preparation on the part of the students this may be interpreted as the vector equation of the straight line passing through A and B. In particular, with respect to the Cartesian system with A being the point (x_1, y_1, z_1), B$(x_2, y_2, z_2)$and P(x, y, z), the two-point form of the line

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$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

can be easily obtained. At this juncture students may be asked to write down the direction number of the vector $\vec{b}-\vec{a}$ prior to the smooth generalization of the two-point form into

(i) symmetrical form

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ and}$$

(ii) parametric form

$x = x_1 + \ell$
$y = y_1 + m$
$z = z_1 + n$

where $\ell : m : n$ stands for the direction number of the line.

As a continuation, the equation of the plane having normal in the direction $\ell : m : n$ and passing through (x_1, y_1, z_1) can be introduced as an application of dot product:

$$\ell(x-x_1)+m(y-y_1)+n(z-z_1)=0$$

In this connection the general equation of a plane Ax + By + Cz + D = 0 should be introduced as a supplement with the following properties introduced.

(i) the direction ratios of the normal to the plane is A: B: C.

(ii) the perpendicular distance of the point P(x', y', z') to the plane is given by $\frac{Ax'+By'+Cz'+D}{\pm\sqrt{A^2+B^2+C^2}}$, where the sign is chosen so as to make the expression positive.

(iii) the angle θ between two planes

$\pi_1 : A_1x+B_1y+C_1z+D_1=0$ and

$\pi_2 : A_2x+B_2y+C_2z+D_2=0$ is

given by $$\cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}$$

(iv) $\pi_1 // \pi_2$ if and only if $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$

$\pi_1 \perp \pi_2$ if and only if $A_1A_2+B_1B_2+C_1C_2=0$

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(v) the equation of the planes bisecting the angles between two planes π_1 and π_2 are

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2+B_1^2+C_1^2}}=\pm\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

Following the acquisition of the general knowledge of lines and planes, teachers may lead the students to appreciate the fact that

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases} \text{ represents}$$

the line of intersection of the planes π_1 and π_2 (if not parallel) and the direction ratios of the line can be found by

$$\begin{vmatrix}B_1 & C_1\\B_2 & C_2\end{vmatrix}:\begin{vmatrix}C_1 & A_1\\C_2 & A_2\end{vmatrix}:\begin{vmatrix}A_1 & B_1\\A_2 & B_2\end{vmatrix}$$

Furthermore the following properties between a line L with direction ratios p : q : r and a plane π: Ax + By + Cz + D = 0 should be discussed

(i) $L//\pi$ iff $Ap+Bq+Cr=0$

(ii) $L\perp\pi$ iff $\frac{A}{p}=\frac{B}{q}=\frac{C}{r}$

(iii) the angle θ made between L and π is given by

$$\sin\theta=\left|\frac{Ap+Bq+Cr}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{p^2+q^2+r^2}}\right|$$

The conditions for two lines to be coplanar should be also studied i.e. two lines are coplanar if and only if they intersect or are parallel.

Suppose L_1 is $\frac{x-a_1}{p_1}=\frac{y-b_1}{q_1}=\frac{z-c_1}{r_1}$

L_2 is $\frac{x-a_2}{p_2}=\frac{y-b_2}{q_2}=\frac{z-c_2}{r_2}$,

L_1 and L_2 are coplanar iff $\begin{vmatrix}a_1-a_2 & p_1 & p_2\\b_1-b_2 & q_1 & q_2\\c_1-c_2 & r_1 & r_2\end{vmatrix}=0$

Throughout this sub-unit, teachers are encouraged to apply vector approach as far as possible in deducing the above-mentioned properties or results. In particular the use of dot product to find the projection of a vector $\vec{p}$ along a vector $\vec{r}$ and the use of cross product to evaluate the area of triangle with vertices given should be explained.

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