Unit B3: Differentiation

Objective: (1) To acquire different techniques of differentiation.
(2) To learn and acquire techniques to find higher order derivative.
(3) To understand the intuitive concept of Rolle's Theorem and Mean Value Theorem.

Detailed Content	Time Ratio	Notes on Teaching
3.1 Fundamental rules for differentiation	4	As a continuition, the following rules should be taught: (1) $\frac{d}{dx}(k)=0$, where k is a constant (2) $\frac{d}{dx}(x^r)=rx^{r-1}$, where r is real (3) $\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$ (4) $\frac{d}{dx}[kf(x)]=k\frac{d}{dx}f(x)$, where k is a constant (5) $\frac{d}{dx}[f(x)g(x)]=g(x)\frac{d}{dx}f(x)+f(x)\frac{d}{dx}g(x)$ (product rule) (6) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{g(x)^2}$ $g(x)\neq 0$ (quotient rule) Proofs of the above rules should be mentioned or presented as a form of practice in order to strengthen students' mastery of the concept and skill. From (3) to (6), the existence of the derivatives of f(x) and g(x) should be emphasized. Regarding (2), a proof for r being integral will be enough while for the general case r being real the proof may be provided at a later stage till the students have learnt Chain rule. Typical examples in using the above rules to obtain derivative of various common functions should be done for illustration.
3.2 Differentiation of trigonometric functions	2	Differentiation of the following functions should be taught: 1. sinx 2. cosx 3. tanx 4. cosecx 5. secx 6. cotx Students may be encouraged to do the proof themselves under teachers' supervision and, in particular, they should be reminded to derive the results for (4) to (6) using the quotient rule.
3.3 Differentiation of composite functions and inverse functions	4	For a composite function y = f[g(x)], the derivative is obtained through the chain rule $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$ or = f'(t)g'(x) with t = g(x) For the inverse function $x = f^{-1}(y)$ of y = f(x), the derivative is obtained through $\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$ It is suggested that examples like $\frac{d}{dx}(\sin^{-1}x)$, $\frac{d}{dx}(\cos^{-1}x)$, $\frac{d}{dx}(\tan^{-1}x)$ and $\frac{d}{dx}(x^{-n})$, $\frac{d}{dx}(x^{\frac{1}{n}})$ with n being positive integer may be used for illustration.
3.4 Differentiation of implicit functions	2	It is often necessary to differentiate a function defined implicitly by F(x, y) = 0. This is done by differentiating both sides of the given equation with respect to the independent variable x and applying the rules mentioned above. Various illustrating examples should be included to enrich the discussion. The following are some suggestions. (i) If $x\cos y^3+y\sin 2x=1$, find $\frac{dy}{dx}$ (ii) Given $2x^2-y^2+12x-2y+3=0$, find $\frac{dy}{dx}$ at the point (2, 5). (iii) Find $\frac{dy}{dx}$ for $\cos(x^2-y^2)=xy$.

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Detailed Content	Time Ratio	Notes on Teaching
3.5 Differentiation of parametric equations	2	A parametric representation of a function y = f(x) is given by x = u(t) and y = v(t). Hence y can be expressed as a composite function of the parameter t in the form y = f(u(t)). By applying the chain rule for differentiation, the result $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$ and hence $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ or $f'(x)=\frac{v'(t)}{u'(t)}$ can be obtained. It should be clarified that in this derivation it is assumed that u(t) and v(t) are differentiable and $u'(t)\neq 0$. Typical examples for illustration include finding $\frac{dy}{dx}$ for the following functions: (i) the ellipse x = acost, y = bsint (ii) the cycloid. x = a(t-sint), y = a(1 -cost)
3.6 Differentiation of logarithmic and exponential functions	6	The following rules should be taught and their proofs may be provided with the suggested approach. 1. $\frac{d}{dx}(\ell n\, x)=\frac{1}{x}$ (using $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$) 2. $\frac{d}{dx}e^x=e^x$ (using $\frac{d}{dx}(\ell n\, y)=\frac{1}{y}$ and chain rule, where $y=e^x$, or applying the rule about the derivative of inverse function) 3. $\frac{d}{dx}(\log_a x)=\frac{1}{x\,\ell n\, a}$ 4. $\frac{d}{dx}(a^x)=a^x\ell n\, a$ Examples provided should include functions of the types like e^{x^3} and $\log_a\sqrt{x^2+1}$. (N.B. At this juncture the proof for the formula $\frac{d}{dx}x^n=nx^{n-1}$ when n is rational and when n is real may be mentioned for the sake of completeness.)

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Detailed Content	Time Ratio	Notes on Teaching
		Teachers should also highlight some common applications of logarithmic differentiation as follows: when y is a complicated function of x and especially when it involves a variable as index, the value of $\frac{dy}{dx}$ may sometimes be more easily obtained by logarithmic differentiation. Typical examples of this kind include functions like $y = x^x$ and $y=\frac{(x+a)(x+b)}{(x+c)(x+d)}$.
3.7 Higher order derivatives and Leibniz's Theorem	5	The definition of higher order derivatives and the symbols f"(x), $f^{(n)}(x)$, $\frac{d^ny}{dx^n}$ should be introduced. Also the abilities to find higher order derivatives of functions given in parametric form and to apply the Leibniz's Theorem, viz $\frac{d^n}{dx^n}(uv)=\sum_{r=0}^{n}C_r^n\,u^{(r)}v^{(n-r)}$ are expected. Students may attempt to prove the theorem by mathematical induction. Examples showing the use of Leibniz's Theorem in obtaining relations involving higher order derivatives especially of implicit functions should be illustrated. Examples of this kind include 1. Find the n^{th} derivatives of $\cos^2 x\sin x$ and $x^3\cos x$. 2. Let $f(x) = \tan^{-1}x$, show that $(1 + x^2) f''(x) + 2x f'(x) = 0$ and hence obtain the n^{th} derivative of f(x) for x = 0. 3. Given $f(x)=\frac{x^3}{x^2-1}$ show that $f^{(n)}(0)=\begin{cases}0 \text{ if n is even}\\ -n! \text{ if n is odd}\end{cases}$ where n is an integer and $n\geq 3$
3.8 The Rolle's Theorem and Mean Value Theorem	3	The intuitive concept of the Rolle's Theorem and Mean Value Theorem as well as their geometrical interpretation should be taught. For abler students the proof may be mentioned. Simple and straightforward applications of the theorems are expected. The following examples may be considered:

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Detailed Content	Time Ratio	Notes on Teaching
		1. If $f'(x) = 0$ for all x in an interval, then f(x) is constant in that interval. 2. If $f'(x) = g'(x)$ for all x in an interval, then f(x) and g(x) differ in that interval by a constant. 3. Prove that if $\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$ then the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root between 0 and 1
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Unit B4: Application of Differentiation

Objective: (1) To learn and to use the L' Hospital's Rule.
(2) To learn the applications of differentiation.

Detailed Content	Time Ratio	Notes on Teaching
4.1 The L' Hospital's Rule	4	Limits having the following indeterminate forms should be introduced: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞ Accompanying examples illustrating the type mentioned are highly desirable. The L' Hospital's Rule $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ for the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ should be taught in the first place. The examples that follow may be considered: (1) $\lim\limits_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$ (2) $\lim\limits_{x \to a^+} \frac{\ell n \sin(x-a)}{\ell n \tan(x-a)}$ Teachers should emphasize that $\frac{f'(x)}{g'(x)}$ should be simplified before taking limit and the process can be repeated until $\lim\limits_{x \to a} \frac{f^{(m)}(x)}{g^{(m)}(x)}$ is obtained in a non-indeterminate form. As for the other indeterminate forms, examples should be worked out showing that they can be expressed in the determinate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that the rule may be applied. The following examples may be considered: (1) $\lim\limits_{x \to 0} (\frac{1}{x} - \cot x)$ (2) $\lim\limits_{x \to 0^+} x^x$ (3) $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ The proof of the L' Hospital's Rule is not required.

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