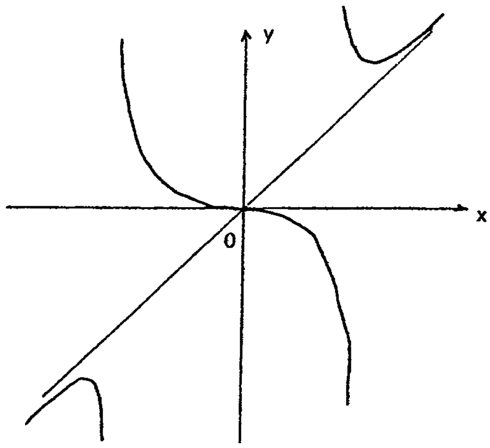
Detailed Content	Time Ratio	Notes on Teaching
		5. $y = \dfrac{x^3}{x^2 - 1}$ 
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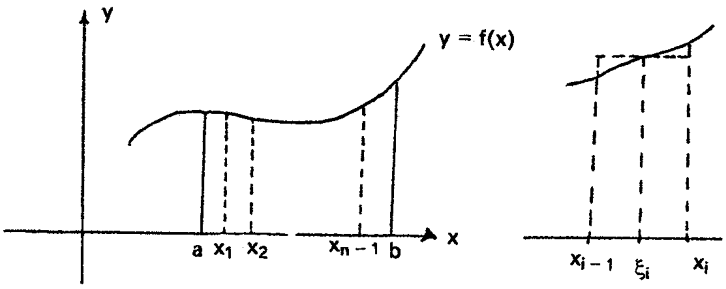
Unit B5: Integration

Objective: (1) To understand the notion of integral as limit of a sum.
(2) To learn some properties of integrals.
(3) To understand the Fundamental Theorem of Integral Calculus.
(4) To apply the Fundamental Theorem of Integral Calculus in the evaluation of integrals.
(5) To learn the methods of integration.
(6) To acquire the first notion of improper integral.

Detailed Content	Time Ratio	Notes on Teaching
5.1 The Riemann definition of integration	5	The theory of the definite integral can be presented in two distinct ways, according as we adopt the geometrical approach or the analytical approach. In the former, the idea of area is presumed, while in the latter the notion of the definite integral as the limit of an algebraic sum without any appeal to geometry is employed. Teachers should determine their choices and sequences of teaching according to the needs of their students. Teachers may start with a function f(x) ≥ 0 for easy understanding and the following simplified version of an intuitive approach is for reference: Let the function f(x) ≥ 0 in the interval [a, b] and therein let the graph of y = f(x) be finite and continuous. 

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Detailed Content	Time Ratio	Notes on Teaching

Partition [a, b] into n subintervals by points x_0, x_1, x_2,…, x_n such that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ and let Δxi denotes $x_i - x_{i-1}$ and ξ_i be an arbitrary point in $[x_{i-1}, x_i]$. The area of the region bounded by the curve y = f(x), the ordinates x = a and x = b and the x-axis can be approximated by the sum

$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$. Moreover, when n increases and max $(\Delta xi) \to 0$,

the value of area can be found and such limit of sum is defined as the definite integral of f(x) from x = a to x = b and it is denoted by

$$\int_a^b f(x)dx \quad \text{i.e.} \quad \int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \max(\Delta x_i)\to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

In the notation,

f(x) is called the integrand; a is called the lower limit; b is called the upper limit and the sum is called the Riemann sum.

Teachers should then generalize the discussion to obtain the definition of Riemann sum of a general function f(x).

During the discussion with students, the following points should be highlighted:

(1) The partition of [a, b] into subintervals is arbitrary;

(2) $\xi_i \in [x_{i-1}, x_i]$ is arbitrary

(3) The definition of the definite integral as the limit of sum presupposes that a < b. Its value when a > b is defined by

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$ and when a = b by

$$\int_a^a f(x)\,dx = 0$$ (N.B. These results may become theorems if definite integrals are defined by means of the function F(x); for

F(b) – F(a) = –[F(a) – F(b)]

F(a) – F(a) =0. Illustrating examples embellishing the verbal presentation of the teachers should be worked out to help students substantiate their understanding. The following examples are for reference.

Detailed Content	Time Ratio	Notes on Teaching

Example 1:

$$\int_a^b e^x\,dx$$

Consider equal intervals $\Delta x_i = \frac{b-a}{n} = h$ (say), then x_0 = a, x_1 = a + h, …, x_{i-1} = $a + (i-1)h$. Choose ξ_i be x_{i-1} i.e. $\xi_i = a + (i-1)h$. As max $\Delta x_i = \Delta x_i = h$,

$$\int_a^b e^x\,dx = \lim_{h\to 0}\sum_{i=1}^{n} e^{\xi_i}h = \lim_{h\to 0} h\sum_{1}^{n} e^{a+(i-1)h} = \lim_{h\to 0} he^a\sum_{1}^{n} e^{(i-1)h}$$

$$= \lim_{h\to 0} he^a\frac{(e^{nh}-1)}{(e^h-1)} = \lim_{h\to 0} he^a\frac{(e^{b-a}-1)}{(e^h-1)} = \lim_{h\to 0} h\frac{(e^b-e^a)}{(e^h-1)}$$

$$= (e^b-e^a)\lim_{h\to 0}\frac{h}{(e^h-1)} = (e^b-e^a)\lim_{h\to 0}\frac{1}{e^h} = e^b - e^a .$$

Example 2:

$$\int_a^b x^m\,dx , m \neq -1.$$

Consider n intervals such that $x_0 = a$, $x_1 = ar$,…, $x_i = ar^i$, $x_n = ar^n = b$. When $n \to \infty$, we have $b = ar^n \Leftrightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$ so that $r \to 1$, and max $\Delta x_i = \Delta x_n = x_n - x_{n-1} = ar^n - ar^{n-1} = ar^n(1-r^{-1}) = b(1-r^{-1}) \to 0$

Choose $\xi_i = x_{i-1} = ar^{i-1}$

$$\int_a^b x^m dx = \lim_{r\to 1}\sum_{i=1}^{n}(ar^{i-1})^m(ar^i - ar^{i-1})$$

$$= \lim_{r\to 1}\sum_{i=1}^{n} a^{m+1}r^{(m+1)(i-1)}(r-1)$$

$$= \lim_{r\to 1} a^{m+1}(r-1)\cdot\frac{r^{(m+1)(n)}-1}{r^{m+1}-1}$$

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Detailed Content	Time Ratio	Notes on Teaching
		$= \lim_{r\to 1} a^{m+1}(r^{(m+1)n}-1)\cdot\dfrac{r-1}{r^{m+1}-1}$ $= \lim_{r\to 1} (b^{m+1}-a^{m+1})\cdot\dfrac{r-1}{r^{m+1}-1}$ $= (b^{m+1}-a^{m+1})\cdot\lim_{r\to 1}\dfrac{1}{(m+1)r^m}$ $= \dfrac{b^{m+1}-a^{m+1}}{m+1}$ Last but not least, teachers should also elaborate on (i) If f(x) is continuous on [a, b], then f(x) is integrable over [a, b] (ii) If f(x) is bounded and monotonic in [a, b], then f(x) is integrable over [a, b].
5.2 Simple properties of definite integrals	4	Teachers may help their students derive the following results from the definition. (1) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, k being a constant (2) $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x)+g(x)]\,dx$ (3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any point inside or outside the interval [a, b]. (4) If f(x) ≥ g(x) for all values of x in [a, b], then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ (5) If $\lvert f(x)\rvert \le \phi(x)$ for all values of x in [a, b], then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le \int_a^b \phi(x)\,dx$. In particular, (a) if $\phi(x) = \lvert f(x)\rvert$, then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le \int_a^b \lvert f(x)\rvert\,dx$ (b) if $\phi(x) = M$, M being constant, then $\left\lvert\int_a^b f(x)\,dx\right\rvert \le M(b-a)$.

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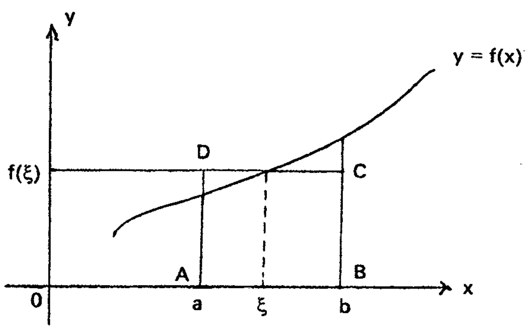
Detailed Content	Time Ratio	Notes on Teaching
		Simple and straightforward applications like the following may be discussed with the students: (1) If f(x) is positive and monotonic increasing for x > 0, prove that $f(n-1) \le \int_{n-1}^{n} f(x)\,dx \le f(n)$ (2) $\left\lvert \dfrac{1}{n}\int_0^1 \dfrac{\sin nx}{1+x^2}\,dx \right\rvert \le \dfrac{\pi}{4n}$
5.3 The Mean Value Theorem for Integrals	2	A simplified version of the theorem is advisable, viz If f(x) is continuous on [a, b], then there exists a number ξ in (a,b) such that $\int_a^b f(x)\,dx = f(\xi)\cdot(b-a)$ The idea conveyed can easily be visualized through the accompanying diagram. Students should find no difficulty to understand the intrinsic meaning of f(ξ) (b – a) being the area of the rectangle ABCD. 

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Detailed Content	Time Ratio	Notes on Teaching
		If a more formal proof is desirable, it can be furnished by using the properties mentioned in 5.2 together with the properties of continuous function and in particular, the Intermediate Value Theorem.
5.4 Fundamental Theorem of Integral Calculus and its application to the evaluation of integrals	4	The First Fundamental Theorem of Integral Calculus, viz Let f(x) be continuous on [a, b] and let F(x) be defined by $F(x)=\int_a^x f(t)\,dt, a \le x \le b$ then (i) F(x) is continuous in [a, b] (ii) F(x) is differentiable in (a, b) and $\frac{d}{dx}F(x)=f(x)$ or the simplified version If f(x) is continuous, then the function $F(x)=\int_a^x f(t)\,dt$ is differentiable and its derivative is equal to the value of the integrand at the upper limit of integration i.e. F'(x) = f(x). This should be discussed thoroughly with the students and students may be, under the supervision of their teachers, led to prove the Theorem using the Mean Value Theorem for Integral Calculus. (N.B. Teachers should, immediately following this theorem, elaborate on the results follow: (1) the function F(x) whose derivative is equal to the integrand f(x) is called a primitive of f(x). (2) for two such primitives F(x) and G(x) of the same integrand, the derivative of F(x) − G(x) is identically zero, so F(x) − G(x) is constant.) Regarding The Second Fundamental Theorem of Integral Calculus, teachers may again assist their students in the derivation. The version that follows is for consideration: Let f(x), and F(x) be continuous in [a, b]; if $\frac{d}{dx}F(x)=f(x)$ for $a < x < b$, then for $a < x \le b$, $\int_a^x f(t)\,dt = F(x)-F(a)$ and, in particular $\int_a^b f(x)\,dx = F(b)-F(a)$.

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Detailed Content	Time Ratio	Notes on Teaching
		Some enlightening examples in evaluating definite integrals by taking it as an infinite sum in the first place and then by finding its primitive as an alternative solution should be worked out so that students' overall understanding on the theorems taught can be strengthened and hence their awareness of the alternative approach in evaluating integrals through the reverse process of differentiation can be promoted. Teachers may start with simpler ones like the following $\int_a^b x^2\,dx = \frac{b^3}{3}-\frac{a^3}{3}$ and end up with other interesting applications like (1) By considering $f(x)=\frac{1}{x}$ in interval [1,2], the result that $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n} \to$ ($\ell n\,2$ as $n \to \infty$) can be established. (2) By considering $f(x)=\frac{1}{1+x^2}$ over (0,1), one can show that as $n \to \infty$, $n\sum_{r=2}^{n}\frac{1}{r^2+n^2}=\frac{\pi}{4}$.
5.5 Indefinite integration	6	As a continuation, this section is devoted to focus students' attention to the mechanical process of finding .primitive as an alternative approach to evaluate definite integrals. The notation $\int f(x)dx$ representing the indefinite integral of f(x) should be introduced in the sense that If $\frac{d}{dx}F(x)=f(x)$ holds, then F(x) is said to be an Indefinite Integral of f(x) and is denoted by $F(x)=\int f(x)dx$. Teachers should also point out that indefinite integral of f(x) is not unique and that if F(x) is an indefinite integral of f(x), then F(x) +c where c is a constant, is another, treating $\int f(x)dx$ as a primitive of f(x). Students are expected to be able to apply the following formulae for evaluating indefinite integrals. As a matter of fact, they can be encouraged to derive some or all of them.

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Detailed Content	Time Ratio	Notes on Teaching
		(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ (2) $\int \frac{dx}{x} = \ell n(x) + c$ (3) $\int e^x dx = e^x + c$ (4) $\int a^x dx = a^x \ell na + c$ (5) $\int \sin x \, dx = -\cos x + c$ (6) $\int \cos x \, dx = \sin x + c$ (7) $\int \sec x \tan x \; dx = \sec x + c$ (8) $\int \sec^2 x \, dx = \tan x + c$ (9) $\int \operatorname{cosec} x \cot x \; dx = -\operatorname{cosec} x + c$ (10) $\int \operatorname{cosec}^2 x \; dx = -\cot x + c$ (11) $\int \tan x \; dx = \ell n\lvert\sec x\rvert + c$ (12) $\int \cot x \; dx = \ell n\lvert\sin x\rvert + c$ (13) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$ (14) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ Teachers may also remind the students of the following properties (1) $\int kf(x)\,dx = k\int f(x)\;dx$, k is a constant. (2) $\int [f(x)+g(x)]\,dx = \int f(x)\;dx + \int g(x)\,dx$ Students should be encouraged to have adequate practices on a sufficient variety of indefinite integrals in order to testify their mastery of the elementary manipulation to facilitate smoother acquisition of the forthcoming techniques.

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Detailed Content	Time Ratio	Notes on Teaching
5.6 Method of integration (A) Method of Substitution	8	It is suggested that the substitution formula $\int f(u)\,du = \int f[g(x)]\,g'(x)\,dx$ need not be proved rigorously, however teachers are advised to start with simpler and obvious ones like $\int \frac{dx}{x+1}, \int (x+1)^{10}\,dx, \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$, $\int \sin^5 x \cos x \, dx, \int \frac{dx}{x\,\ell n x}$ etc. In some integrals when g'(x) does not readily appear, g(x) has to be guessed such as the cases $\int \frac{1+e^x}{1-e^x}dx$, $\int \sqrt{1-x^2}\,dx$ etc, students have to develop the technique through a lot of relevant practices like the following (1) $\int_0^{\pi/2} \frac{dx}{2+\sin x}$ (2) $\int \frac{dx}{\cot x + \operatorname{cosec} x}$ (3) $\int \frac{dx}{\sqrt{e^x - 1}}$ (let $u = e^x$) (4) $\int \frac{e^{2x}dx}{\sqrt[4]{e^x+1}}$ (let $u = e^x + 1$) (5) $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$ where $b > a$ (let $x = a\cos^2\theta + b\sin^2\theta$) (6) $\int_0^1 \frac{\ell n(1+x)}{1+x^2}dx$ (let $x = \tan\theta$) The following useful results should also be discussed with students with supporting examples for illustration: (1) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and, in particular $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

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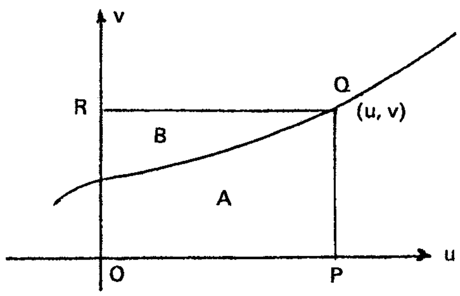
Detailed Content	Time Ratio	Notes on Teaching
		(2) If $f(x)=f(a-x)$, then $\int_0^a xf(x)\,dx=\frac{a}{2}\int_0^a f(x)\,dx$ and in particular $\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$ (3) If f(x) is periodic with period w then $\int_a^{a+w} f(x)\,dx=\int_0^w f(x)\,dx$ (4) If f(x) is an even function, then $\int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx$ (5) If f(x) is an odd function, then $\int_{-a}^a f(x)\,dx=0$ (6) $\int_0^{\pi/2} f(\cos x)\,dx=\int_0^{\pi/2} f(\sin x)\,dx=\frac{1}{2}\int_0^\pi f(\sin x)\,dx$ Related examples suggested for consideration are as follows: (1) $\int_0^\pi \frac{\cos^3 x}{\sin x+\cos x}\,dx$ (2) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$ (3) Show that $\int_0^a x^m(a-x)^n dx=\int_0^a x^n(a-x)^m dx$, and hence evaluate $\int_0^8 x^2\sqrt[3]{8-x}\,dx$ (4) $\int_{-\pi}^{\pi} x^4 \sin x dx$ (5) Show that $\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx$ and hence evaluate the integral. (6) Show that $\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}\,dx=\pi/4$.

Detailed Content	Time Ratio	Notes on Teaching
(B) Integration by Parts	3	The integration by parts formula $\int f(x)\,g'(x)\,dx=f(x)\,g(x)-\int g(x)\,f'(x)\,dx$ or $\int udv=uv-\int vdu$ can readily be proved using the intuitive geometrical approach, like  The diagram suggests an informal geometrical interpretation of the formula: Area of region A can be represented by $\int vdu$; Area of region B by $\int udv$; Area of OPQR by uv and hence the formula is readily depicted. Typical examples for illustration include $\int xe^x dx$, $\int x\sin x dx$ and $\int \ell n\, x\, dx$ With the combination of the method of substitution and integration by parts formula, students are able to handle many different kinds of integrals like (1) $\int e^{ax}\cos bx\,dx$ (2) $\int \tan^{-1}x\,\ell n\,(1+x^2)\,dx$ (3) $\int \left(\frac{1}{x}+\frac{1}{x^2}\right)\ell n\,x\,dx$ (4) $\int_0^1 \sin^{-1} x dx$

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Detailed Content	Time Ratio	Notes on Teaching
		(5) $\int_0^1 x\tan^{-1}x\,dx$ (6) $\int_0^{\pi/2} \frac{x\cos x\sin x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$
(C) Reduction Formula	5	Reduction formula is used to express the integral of the general member of a class of functions in terms of simpler member(s) of the class. The reduction formula is generally obtained by applying the method of integration by parts. It is quite extensively used in the integration of trigonometric functions. Typical examples for consideration are as follows: (1) Let I_n denote $\int_0^{\pi/4}\tan^n x\,dx$, show that $I_n=\frac{1}{n-1}-I_{n-2}, n\geq 2$ hence Evaluate I_4. (2) Let $I_n=\int\frac{dx}{(x^2+a^2)^n}$, obtain a reduction formula for I_n and then evaluate $\int_0^a\frac{dx}{(x^2+a^2)^3}$. (3) If $I_n=\int x^n e^{x^2}\,dx$, show that $I_n=\frac{1}{2}x^{n-1}e^{x^2}-\frac{1}{2}(n-1)I_{n-2}$ for n > 2.
(D) Integration by Partial Fractions	4	Integration of rational algebraic functions may be achieved by splitting the expressions into partial fractions. There are four types of fractions in general: $\int\frac{Ldx}{ax+b}, \int\frac{Ldx}{(ax+b)^r}, \int\frac{Lx+M}{(ax^2+bx+c)}dx$ and $\int\frac{Lx+M\,dx}{(ax^2+bx+c)^r}$ Students should be able to handle the first three types without significant difficulty while for the last type, the application of reduction formula is required. Some examples suggested for discussion are (1) $\int_{-1}^1\sqrt{\frac{x+3}{x+1}}\,dx$ (2) $\int\left(\frac{x}{x^2-3x+2}\right)^2 dx$

Detailed Content	Time Ratio	Notes on Teaching
		(3) Let $I_n=\int\frac{dx}{(1-x^4)^n}$, show that $4nI_{n+1}=(4n-1)I_n+\frac{x}{(1-x^4)^n}, n\geq 1$ and evaluate $\int\frac{x^8}{(1-x^4)^4}dx$ However, it is worthwhile for students to note that some integrals cannot be expressed in elementary terms, like $\int e^{-x^2}dx$, $\int\sin(x^2)\,dx, \int\frac{\sin x}{x}dx$, … etc.
5.7 Improper integrals	4	The first notion of improper integral is to be introduced and students are expected to be able to recognize improper integral of the first type viz, $\lim_{b\to\infty}\int_a^b f(x)\,dx$ or $\lim_{a\to-\infty}\int_a^b f(x)\,dx$ which may be simply denoted by $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ and improper integral of the second type, viz when $\lim_{x\to a}f(x)=\infty$ $\int_a^b f(x)\,dx=\lim_{h\to 0^+}\int_{a+h}^b f(x)\,dx$ and when $\lim_{x\to b}f(x)=\infty$, $\int_a^b f(x)\,dx=\lim_{h\to 0^+}\int_a^{b-h}f(x)dx$. Typical examples of the first type: (1) $\int_0^\infty\frac{dx}{1+x}$ (2) $\int_{-\infty}^1\frac{dx}{x^2}$ Teachers are advised to put forth the example $\int_1^\infty\frac{dx}{x}$ and pinpoint that this is not an improper integral as the limit does not exist.

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Detailed Content	Time Ratio	Notes on Teaching
		Typical examples of the second type $\int_0^1 \frac{dx}{\sqrt{x}}$ and $\int_{-1}^1 \frac{dx}{\sqrt{1-x}}$. Likewise, teachers may use $\int_0^1 \frac{dx}{\sqrt{x}}$ as illustration that this again is not an improper integral as the limit does not exist either.
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Unit B6: Application of Integration

Objective: (1) To learn the application of definite integration in the evaluation of plane area and, arc length, volume of solid of revolution and area of surface of revolution.
(2) To apply definite integration to the evaluation of limit of sum.

Detailed Content	Time Ratio	Notes on Teaching
6.1 Plane area	5	As a sequel to the definition of definite integral, the area bounded by a curve y = f(x), the ordinates x = a and x = b and the x-axis can be evaluated in the following ways depending on the nature of the function (being above or below the x-axis): case (i) when y = f(x) is continuous and non-negative in [a, b], then the area so bounded is given by $\int_a^b f(x)\,dx$ case (ii) when f(x) is continuous and non-positive in [a, b], the area is given by $-\int_a^b f(x)\,dx$

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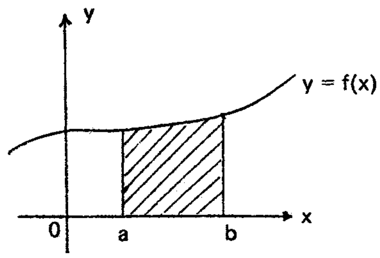
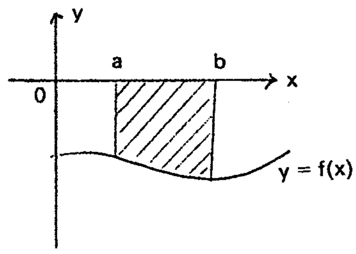


case (i)



case (ii)

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