Detailed Content	Time Ratio	Notes on Teaching
		Typical examples of the second type $\int_0^1 \frac{dx}{\sqrt{x}}$ and $\int_{-1}^1 \frac{dx}{\sqrt{1-x}}$. Likewise, teachers may use $\int_0^1 \frac{dx}{\sqrt{x}}$ as illustration that this again is not an improper integral as the limit does not exist either.
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Unit B6: Application of Integration

and

Objective: (1) To learn the application of definite integration in the evaluation of plane area, arc length, volume of solid of revolution and area of surface of revolution.
(2) To apply definite integration to the evaluation of limit of sum.

Detailed Content	Time Ratio	Notes on Teaching
6.1 Plane area	5	As a sequel to the definition of definite integral, the area bounded by a curve y = f(x), the ordinates x = a and x = b and the x-axis can be evaluated in the following ways depending on the nature of the function (being above or below the x-axis): case (i) when y = f(x) is continuous and non-negative in [a, b], then the area so bounded is given by $\int_a^b f(x)\,dx$ case (ii) when f(x) is continuous and non-positive in [a, b], the area is given by $-\int_a^b f(x)\,dx$

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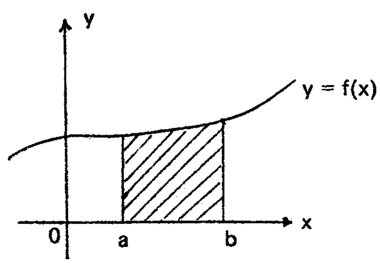
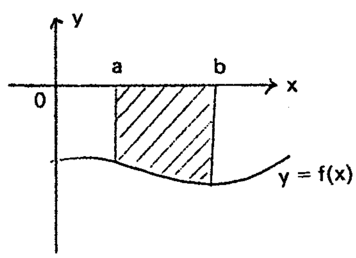


case (i)



case (ii)

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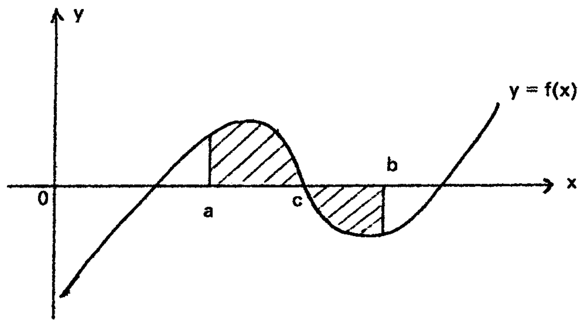
Detailed Content	Time Ratio	Notes on Teaching
		case (iii) when f(x) is continuous and has positive and negative values in [a, b], then the area bounded can be found by following the approach shown in the simplified example that follows.  Area is given by $\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$. Students should be reminded of the minus sign for area enclosed below the x-axis thus they should be encouraged to have a rough sketch of the function so as to obtain a clearer picture.

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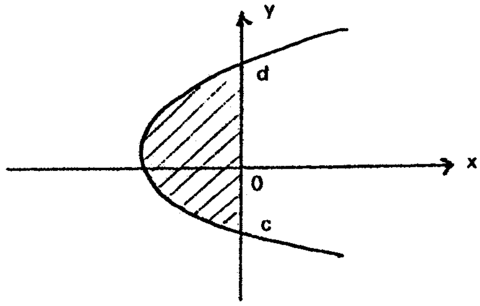
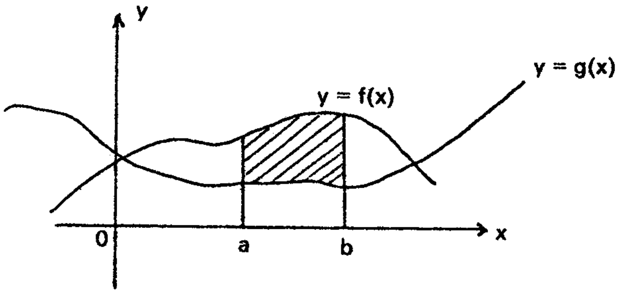
Detailed Content	Time Ratio	Notes on Teaching
		Teachers should also elaborate on the cases where the area enclosed was made with the y-axis like the following diagram.  Area given by $-\int_c^d x\,dy$ For area bounded by two curves, the following approach together with other variations which are illustrated diagrammatically should be discussed thoroughly with students with adequate exemplification.  Area = $\int_a^b [f(x) - g(x)]\,dx$

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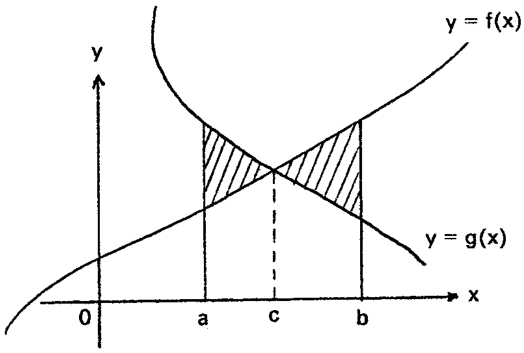
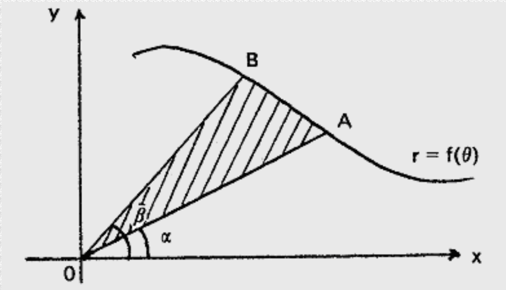
Detailed Content	Time Ratio	Notes on Teaching
		 Area = $\int_a^c [g(x)-f(x)]dx + \int_c^b [f(x)-g(x)]dx$ When the polar equation of the curve is given say, $r = f(\theta)$, then the area bounded by the curved and between the two radii is given by $\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$. 

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Detailed Content	Time Ratio	Notes on Teaching
		When the equation of the curve is given in parametric form say, x = x(t); y = y(t), then the area so enclosed is given by $\frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$ where the parameter of A is t_0 and B is t_1. A good spectrum of worked examples covering some relatively significant variations of the approaches mentioned is highly desirable to enable a better grasp of the skill on the part of the students. The following ones may be considered: (1) Find the area bounded by the parabola $y^2 = 5 - x$ and the line $y = x + 1$. (Note that the area enclosed may be found by integrating with respect to x or to y. Teachers are encouraged to demonstrate both ways.) (2) Show that the area enclosed by the cardioid $r = a(1 + \cos\theta)$ is given by $\frac{3a^2\pi}{2}$. (3) By using the parametric equation $x = a\cos\theta$; $y = b\sin\theta$ of an ellipse, show that the area enclosed is πab. (N.B. Teachers are advised to elaborate on the fact that sometimes the special geometrical properties like symmetry of a figure may help simplify the evaluation process.)
6.2 Arc length	3	The length of arc of the curve y = f(x) between two points on the curve at x = a and x=b is given by $\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx$ If the curve is given in parametric form x = x(t); y = y(t) then tbe arc length of the curve from $t = t_1$ to $t = t_2$ is given by $\int_{t_1}^{t_2}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$ If the curve is given in polar form $r = r(\theta)$, then the arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ is given by $\int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$.

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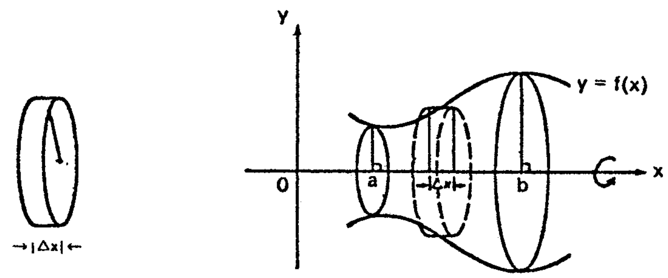
Detailed Content	Time Ratio	Notes on Teaching
		The following suggested examples are for discussion and illustration: (1) Show that the perimeter of the closed curve (asteroid) $x = a\cos^3\theta;\ y = a\sin^3\theta$ is 6a. (Symmetry of the curve about the axes helps) (2) Show that the length of 'the circumference of the cardioid $r = a(1 + \cos\theta)$ is 4a. (3) Find the length of the arc of the curve $x^3 = 8y^2$ from x = 1 to x = 3. (N.B. In choosing curves for illustration, teachers should be well aware of the case of an ellipse and, if deem desirable, may lead a brief discussion with students so as to broaden their perspective on other branches of mathematics studies. A brief account in this respect is suggested for reference as follows: For the ellipse $x = a\sin\theta;\ y = b\cos\theta$ $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\cos2\theta +$ $b^2\sin^2\theta = a^2(1-e^2\sin^2\theta)$ where $e = \frac{b}{a}$ (commonly known as the eccentricity). The arc s measured from an extremity of the minor axis is given by $a\int_0^{\phi}\sqrt{1-e^2\sin^2\theta}\,d\theta$. This integral cannot be expressed in terms of elementary functions in a finite form. It is called an elliptic integral of the second kind denoted by $E(e, \phi)$. For the sake of completeness, teachers may also introduce the elliptic integral of the first kind, viz $\int_0^{\phi}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}}$ which is denoted by $F(e, \phi)$.)
6.3 Volume of revolution	4	It is desirable to have some preliminary discussion with the students on the meaning and formation of solids of revolution while the term axis of revolution should also be introduced so that students may be able to identify solids of revolution and visualize the solids formed when certain segment of curve or region is revolving about certain axis. Teachers may then touch upon the two common methods in finding volume of revolution, viz,

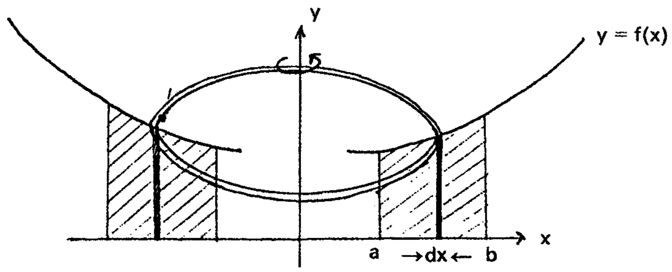
Detailed Content	Time Ratio	Notes on Teaching
		(1) The Disc Method



Teachers should emphasize on the expression of the volume element ΔV or dV that $dV = \pi y^2\,dx$ which is the volume of the disc. The volume of the solid is given by $\pi\int_a^b y^2dx$.

Teachers are also advised to elaborate a bit more on $\pi\int_c^d x^2\,dy$ which is the case when the curve is revolving about the y-axis.

(2) The Shell Method



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Detailed Content	Time Ratio	Notes on Teaching
		In this case the volume element is $2\pi xydx$ and the volume is given by $2\pi\int_a^b xydx$. There are cases in which the solid is formed by revolving about certain lines other than the axes or formed by revolving a region bounded by two curves Thus teachers should elaborate on these cases with the so-called formulae like $\pi\int_a^b [\,[f(x)]^2 - [g(x)]^2]\,dx$ or $\pi\int_c^d [\,[f(y)]^2 - [g(y)]^2]\,dy$ derived for students reference. Adequate illustration is highly recommended. The following are some for reference: (1) Show that the volume generated by rotating the ellipse $x = a\cos\theta; y = b\sin\theta$ about the x-axis is $\frac{4}{3}\pi ab^2$. (N.B. Teachers may request the students to deduce the volume of a sphere of radius r from the given result.) (2) Find the volume of the solid formed by revolving about t h e line x = 2 the region which is Pounded by the curve $y = x^3$, the line x = 2 and the x-axis. (N.B. Teachers are advised to solve this problem using the disc approach as well as the shell approach.)
6.4 Area of surface of revolution	4	The surface area generated by revolving about the x-axis the arc of the curve y = f(x) between x = a and x = b is given by $2\pi\int_a^b yds$ where ds is the element arc length and so the formula usually appears as $2\pi\int_a^b y\sqrt{1+(\frac{dy}{dx})^2}\,dx$ If the said arc length is revolved about the y-axis, the surface area is given by $2\pi\int_c^d xds$ or $2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ where c, and d are respectively the ordinate's of the end points of the arc.

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Detailed Content	Time Ratio	Notes on Teaching
		If the curve is given in polar form with 8 as the independent variable, the area generated is given by $2\pi\int_\alpha^\beta y\frac{ds}{d\theta}d\theta$ where $y = r\sin\theta$ and $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$. Similarly, when in parametric form with the curve given by x = x(t); y = y (t) from $t = t_0$ to $t = t_1$, the area of the surface generated is given by $2\pi\int_{t_0}^{t_1} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ The following examples are suggested for class discussion: (1) Show that the area of the surface generated by revolving about the x-axis an arc of the parabola $y^2 = 4x$ between the origin and the point (4, 4) is $\frac{8\pi}{3}(5\sqrt{5}-1)$. (2) Show that the surface area generated by revolving the cardioid r = a(1 + cosθ) about the initial line is $\frac{32}{5}\pi a^2$. (3) Show that the surface area generated by revolving the cycloid x = a(1 − sinθ); y = a(1 − cosθ) about the x-axis is given by $\frac{64}{3}\pi a^2$. (4) Prove that the area of the surface formed by revolving the asteroid $x = a\cos^3 t$; $y = a\sin^3 t$ about the x-axis is $\frac{12}{5}\pi a^2$.
6.5 Limit of Sum	4	In teaching this interesting application of definite integral, it is advisable to start with some simple and obvious series like $\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}+\frac{n}{n^2}$, and students should be provided with adequate hints so that they manage to associate the limit of sum of the series with the limit of sum leading to the relevant definite integral in a suitable interval and with pertinent partition and most important of all with the appropriate integrand, viz $f(x) = x$ in [0, 1] with partition 0, $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, 1.

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Detailed Content	Time Ratio	Notes on Teaching
		It would not be a difficult task for students to establish the result that $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots+\frac{n}{n^2}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\cdot\frac{1}{n}$ $=\int_0^1 x\,dx$ $=\frac{1}{2}$ Other examples of great mathematical insight like (1) $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n-1}\right)=\int_0^1\frac{dx}{1+x}=\ell n\,2$ (2) $\lim_{n\to\infty}\left(\frac{n}{n^2}+\frac{n}{n^2+1}+\dots+\frac{n}{n^2+(n-1)^2}\right)$ $=\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{4}$ (3) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n(n+1)}}+\dots+\frac{1}{\sqrt{n(2n-1)}}\right)$ $=\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{\sqrt{1+\frac{1}{n}}}+\dots+\frac{1}{\sqrt{1+\frac{n-1}{n}}}\right)$ $=\int_0^1\frac{dx}{\sqrt{1+x}}=2(\sqrt{2}-1)$ may be provided to further their understanding and manipulative technique. The following example is worth discussing as it brings exhilarating result: To find $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$ by transforming it into $\ell n\,y=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\ell n\left(\frac{i}{n}\right)$ where $y=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$ $=\int_0^1\ell n\,x\,dx=-1$ thus

Detailed Content	Time Ratio	Notes on Teaching
	~~20~~ 13	obtain $y=e^{-1}$ (N.B. Teachers may relate this part to Unit B5 in which the idea of Riemann sum expressed as a limit of sum of a series is touched upon.) However, students should be reminded that not all limits of sum can be dealt with using this approach, the harmonic series, $\sum\frac{1}{n}$, is an example.

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