Exemplar 3:

Formula for the Areas of Circles

Objective: To explore the formula for the area of a circle

Key Stage: 3

Learning Unit: Simple Idea of Areas and Volumes

Materials Required: Dynamic Geometry software such as *Geometer's Sketchpad* (later referred as *Sketchpad*) and the file cirarea.gsp

Prerequisite knowledge: The formula for the area of a triangle and the circumference of circle

Description of the Activity:

1. The teacher shows a grid paper to students. On the grid paper, there is a circle with radius *r*. A square *EFGH* is drawn around the circle so that each side of the square touches the circle. Two perpendicular diameters are drawn so that they divide the circle into four equal parts. They also divide the large square *EFGH* into four equal small squares (Fig. 1).

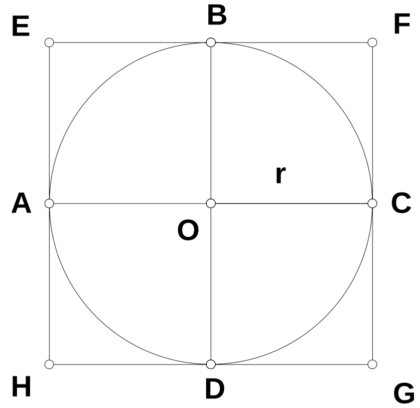

Fig. 1

2. The teacher asks the following questions:
   (a) What is the area of the small square *OAEB*?
   (b) What is the area of the square *EFGH*?
   (c) Is the area of the circle larger or smaller than that of the square *EFGH?* Why?

3. The end points of the diameters are joined as shown in Fig. 2. The teacher then asks students the following questions:
   (a) What is the area of the square *ABCD*?
   (b) Is the area of the circle larger or smaller than that of the square *ABCD*? Why?
   (c) According to the above results, what is the range in which the area of the circle lies?

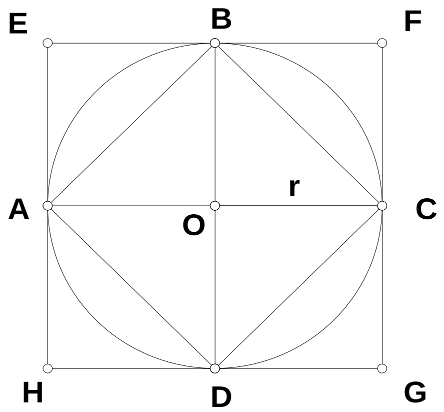

Fig. 2

4. The teacher gives a *Sketchpad* file *cirarea.gsp* to students and asks them to open it (Fig. 3). This program is to approximate the area of a circle by inscribing regular polygons with varying number of sides inside the circle. Students are asked to drag the point "Drag" along the segment and observe how the polygon and the measures change.

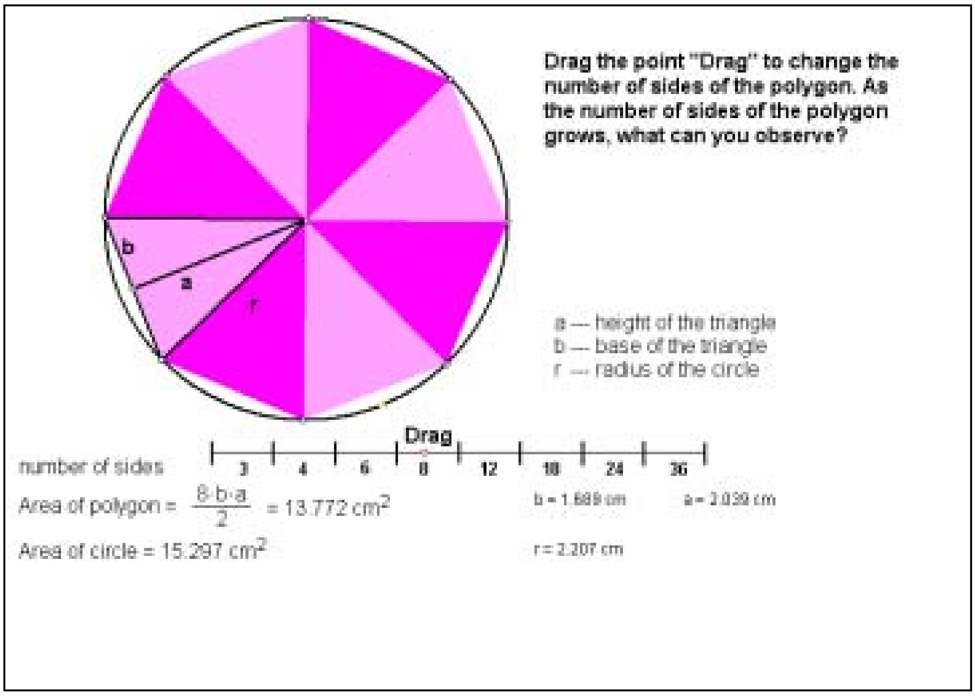

Fig. 3

5. The teacher then asks students the following questions:
   When you drag the point "Drag" to the right,
   (a) what does the polygon change?
   (b) what are the changes in a, b and r? Increase, decrease or no change?
   (c) how does the area of the polygon change when you increase the number of sides of the polygon?

6. Students are guided to discover that as the number of sides of the polygon gets larger, the area of the polygon approaches to the area of the circle.

7. The teacher guides students to derive the formula for the area of a circle by referring to Fig. 3 and by posing the following questions to initiate class discussion:
   (a) What is the area of each triangle? Express it in terms of a and b.
   (b) What is the area of the regular octagon?
   (c) If a regular n-sided polygon is inscribed in the circle instead, express the area of the n-sided polygon in terms of a, b and n.
   (d) As the number of sides of the regular polygon (i.e. the value of n) increases, what does a approach?
   (e) What is the perimeter of the regular n-sided polygon? What does the perimeter of the polygon approach as n increases?
   (f) What is the formula for the circumference of a circle? When n tends to infinity, by using the previous results, rewrite the area of the polygon to represent the area of a circle.

8. The teacher then introduces the stories of Ancient mathematician including Liu Hui[1] who had used this method (named as the Exhaustion Method) to estimate the value of π.

[1] The Chinese name of Liu Hui is 劉徽.

Notes for Teachers:

1. Through this activity, students can approximate the area of a circle by increasing the number of sides of a regular polygon inscribed in the circle. This activity can also enable students to realize and appreciate the past attempts in approximating the value of π .

2. Suggested answers to the questions:

   2 (a): r^2.

   2 (b): $4r^2$.

   2 (c): The area of the circle is smaller than the area of the square *EFGH* since the circle is inscribed in the square.

   3 (a): $2r^2$.

   3 (b): The area of the circle is larger than the area of the square *ABCD* since the square is inscribed in the circle.

   3 (c): $2r^2 <$ area of the circle with radius $r < 4r^2$.

   5 (a): The number of sides of the polygon increases when students drag the point "Drag" to the right.

   5 (b): The values of a, b and r will increase, decrease and remain unchanged respectively.

   5 (c): The area of the polygon will increase and approach to the area of the circle.

   7 (a): $\frac{ab}{2}$.

   7 (b): $4ab$.

   7 (c): $\frac{abn}{2}$.

   7 (d): The radius of the circle, i.e. r.

   7 (e): The perimeter is bn. When n increases, the perimeter approaches the circumference of the circle.

   7 (f): The formula for the circumference of a circle is $2\pi r$. When n tends to infinity, the area of the polygon tends to $\frac{r \cdot 2\pi r}{2}$, i.e. πr^2. Therefore, the area of a circle is πr^2.

3. The teacher may request students to do a project about the history of estimating the value of π.

Reference:

Students can obtain useful information from the following websites, books and articles:

Web sites:

1. http://forum.swarthmore.edu/dr.math/faq/faq.pi.html
2. The following web sites concerning Archimedes' approximation of π:
   http://www.math.utah.edu/~alfeld/Archimedes/Archimedes.html
   http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html
3. For the Buffon's Needle method:
   http://www.angelfire.com/wa/hurben/buff.html
   http://www.mste.uiuc.edu/reese/buffon/buffon.html
   http://www.math.uah.edu/stat/buffon/buffon2.html
4. Others:
   http://gallery.uunet.be/kurtvdb/pi.html
   http://www.geocities.com/hjsmith_geo/download.html#PiW

Books and Articles:

1. Blatner, David (1999). *The Joy of* π. New York: Walker & Co.
2. Cheney, E.W. and Kincaid, D.R. (1999). *Numerical Mathematics and Computing*. 4th Edition. Pacific Grove, California: Brooks/Cole Publishing Company.
3. Schroeder, L. (1974). Buffon's needle problem: An exciting application of many mathematical concepts. In *Mathematics Teacher*, 67(2), pp. 183 – 186. Reston : National Council of Teachers of Mathematics.

Video:

Teacher can find useful information from the following video:

Project Mathematics! ***The Story of Pi*** ©1989 California Institute of Technology, Caltech 1 – 70, Pasadena, CA 91125