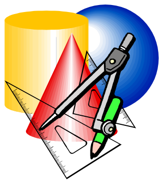


Exemplar 20:

Lines in a Triangle

Objectives: (1) To explore and recognize the concurrent properties of medians and the ratio at which the centroid divides the medians.
(2) To prove that the centroid divides medians in the ratio of 2:1

Key Stage: 3

Learning Unit: Simple Introduction to Deductive Geometry

Materials Required: Dynamic Geometry software such as *Geometer's Sketchpad* (later referred as *Sketchpad*)

Prerequisite Knowledge: (1) Meaning of median
(2) Basic understanding of the conditions of congruence and similarity of triangles

Description of the Activity:

1. Students are asked to explain their understanding of medians of a triangle. The teacher then explains the objectives of the lesson.

2. The Worksheet is distributed to students. Students are asked to construct the medians in a triangle by following the instructions of Construction 1 in the Worksheet. Students who are more competent in using *Sketchpad* are invited to help other students. Students can then use the figure constructed to explore the concurrence of the medians.

3. The teacher asks students to do Investigation 1 in the Worksheet:
   (a) When you construct the third median, do you observe anything special?
   (b) Move the vertices of the triangle, what do you observe?
   (c) Write a conjecture about the medians.

4. After Investigation 1, the teacher can lead the class to discuss the way of intersection of medians in triangles. The concurrence of all medians and the

name of the intersecting point should be concluded with emphasis that this phenomenon is true for any triangle.

5. Students are asked to make other conjectures to the properties of the centroid by dragging the vertices of the triangle. The teacher may ask questions focusing on the lengths if students do not have any idea. Time for students to make guesses should be allowed. After students have an intuitive idea on the ratio at which the centroid divides the medians, students are invited to share their ways of checking the conjectures. After then, students can follow Construction 2 in the lower part of the Worksheet and Investigation 2 of the Worksheet to check their conjectures:

   (a) Calculate the ratios $\frac{AG}{AD}$, $\frac{BG}{BE}$ and $\frac{CG}{CF}$. What do you notice about the ratios? Do these ratios change when you move the triangle?

   (b) Calculate the ratios $\frac{AG}{GD}$, $\frac{BG}{GE}$ and $\frac{CG}{GF}$. Make a conjecture about the way the centroid G divides the medians of a triangle.

6. After Investigation 2, the teacher may ask students to conclude their findings: "The centroid divides the medians in the ratio of 2:1 and the medians are concurrent for any types of triangles". The teacher should help students to conclude the invariance of the ratio quantity. The teacher should remind students that using measurement is only one way of counter-checking their conjectures and guide students to see the limitation of this method.

7. The teacher then guides students to lay out strategies to prove their discoveries. Various approaches may be discussed.

8. For average students, the teacher can ask them to use similarity and congruence of triangle for the proof (see Point 2 of the **Notes for Teachers**, Methods 1 and 2).

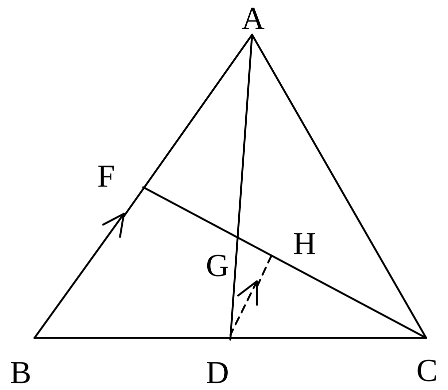
9. Some hints can be given to those students. For instance, for point 2 of Method 1 in the **Notes for Teachers**, the teacher can use the "working backwards" method in posing the following questions to guide students to select the appropriate strategies (see the following figure for reference):

   (a) If we want to prove $AG : GD = 2:1$, which pair of triangles should be considered? Is there sufficient condition to prove the conjecture with focus only on Δ*AFG* and Δ*GDC*? Should we add additional lines to the figure?

(b) If we add a line *DH* parallel to *AB*, which pair of triangles should be considered in proving the ratio? As we want to prove that Δ*DHG* ~ Δ*AFG* in order to justify the ratio, what conditions do we need in proving the similarity of the triangles? Which other pair of triangles can be considered as a link?

(c) If Δ*CDH* and Δ*CBF* are the link, what relation between these triangles can help us to relate the lengths concerned? What is the ratio of *BF*: *DH*? And what is the ratio of *AF* : *HD*?

(d) Can we prove Δ*DHG* ~ Δ*AFG*? What is the ratio of *AG* : *GD*?



10. For more able students, the teacher can request them to use more than one method to carry out the proof (see **Notes for Teachers** Point 2, Methods 1 to 4).

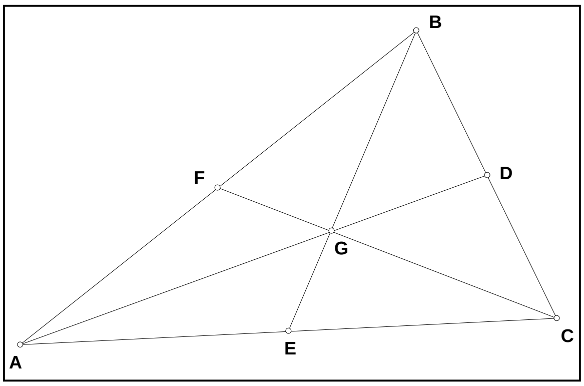
11. The more able students may try to complete the proofs by themselves. Nevertheless, the teacher should give some hints if they have problems.

Worksheet: Exploring the Properties Related to the Medians of a Triangle

You are asked to discover how the medians of a triangle relate to each other. The steps are as follows:

- Use *Sketchpad* and follow the instruction to construct the medians in a triangle.
- Use the sketch drawn to investigate relationships involving medians.
- Write down your answers to the questions in the worksheet.

Construction 1:



(a) Draw ΔABC.
(b) Construct the midpoint of segment *BC*.
(c) Label the midpoint as *D*.
(d) Draw a segment from vertex *A* to midpoint *D*.
(e) Construct the second median to side *AC* in the same manner. Label the midpoint of segment *AC* as *E*.
(f) Label the point of intersection of these two medians as *G*.
(g) Construct the third median to side *AB* in the same manner. Label the midpoint of segment *AB* as *F*.

Investigation 1:

1. When you construct the third median, do you observe anything special?

   ____

2. Move the vertices of the triangle, what do you observe?

   ____

3. Write a conjecture about the medians.

   ____

Construction 2:

(a) Measure the length of the segment *AG*.
(b) Measure the length of the segment *AD*.
(c) Measure the length of the segment *GD*.
(d) Measure the lengths of other medians and their relevant parts in the same manner.

Investigation 2:

1. Calculate the ratios $\frac{AG}{AD}, \frac{BG}{BE}$ and $\frac{CG}{CF}$. What do you notice about the ratios? Does it change when you move the triangle?

   ____

   ____

2. Calculate the ratios $\frac{AG}{GD}, \frac{BG}{GE}$ and $\frac{CG}{GF}$. Make a conjecture about the way the centroid *G* divides the medians of a triangle.

   ____

   ____

Notes for Teachers:

1. Suggested answers to the worksheet：

   Investigation 1

   1. Yes.
   2. No.
   3. The medians of a triangle intersect at a point.

   Investigation 2

   1. $\frac{2}{3}$.
   2. These three ratios are 2 or $\frac{2}{1}$. In other words, we can say that the centroid divides the medians in the ratio of 2 to 1.

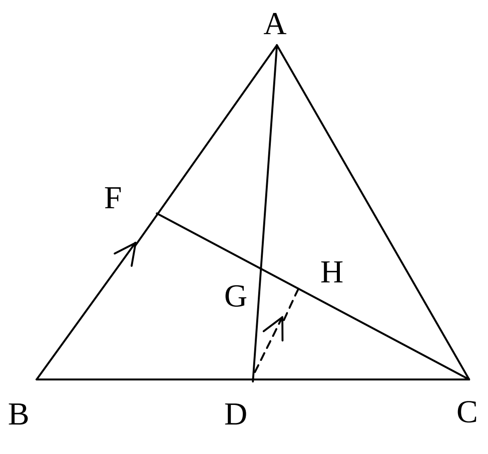
2. Suggested answers to the geometric proofs:

   (1) Prove that the centroid of a triangle divides the medians in the ratio 2:1.

   It should be noted that Method 3 and Method 4 should be introduced after students have learnt the Intercept Theorem and the Mid-point Theorem.

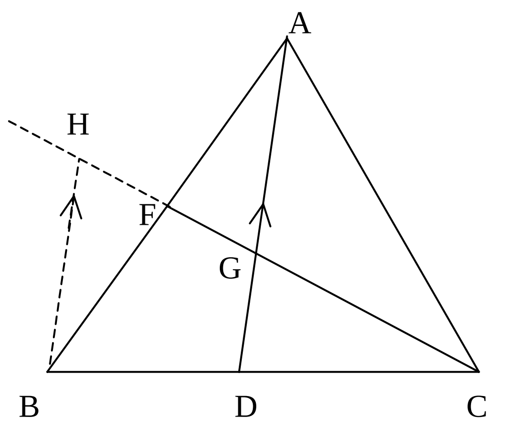
Method 1

Prerequisites: Conditions for similar triangles and the properties of similar triangles

	Key Procedures: 1. Construct medians *AD* and *CF*. 2. Construct *DH* // *BA*. 3. Show that Δ*CDH* ~ Δ*CBF*. 4. Consider the Δ*CDH* and Δ*CBF*, find the ratio *BC* : *DC* and *BF* : *HD*. 5. Show that Δ*DHG* ~ Δ*AFG* and find the ratio *AG* : *GD*.

Method 2

Prerequisites: Conditions for similar and congruent triangles and the properties of similar triangles

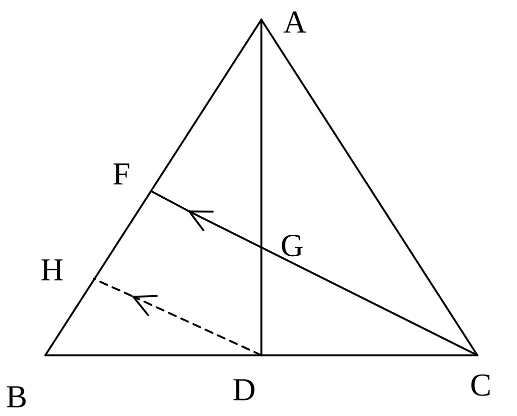


Key Procedures:

1. Construct medians *AD* and *CF*.
2. Produce *CF*.
3. Construct *BH* // *AD*.
4. Show that $\Delta BHF \cong \Delta AGF$.
5. Show that $\Delta CDG \sim \Delta CBH$.
6. Find the ratio *BH* : *DG* and *AG* : *GD*.

Method 3

Prerequisite: Intercept Theorem

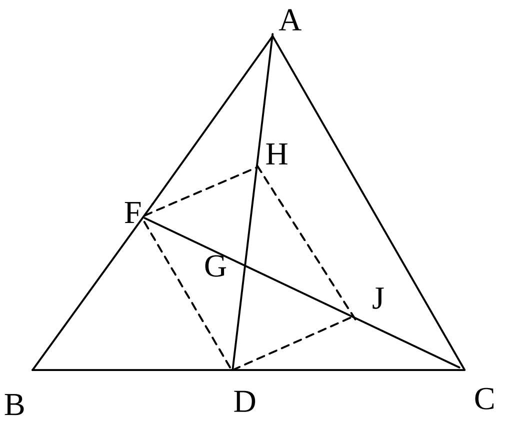


Key Procedures:

1. Construct medians *AD* and *CF*.
2. Construct *DH* // *CF*.
3. Consider the ΔBCF and ΔBDH, apply Intercept Theorem to show that *BH* = *HF*.
4. Find the ratio *AF* : *FH*.
5. Consider the ΔAHD and ΔAFG, apply Intercept Theorem again to get the ratio *AG* : *GD*.

Method 4

Prerequisites : Mid-point Theorem and the properties of parallelogram.

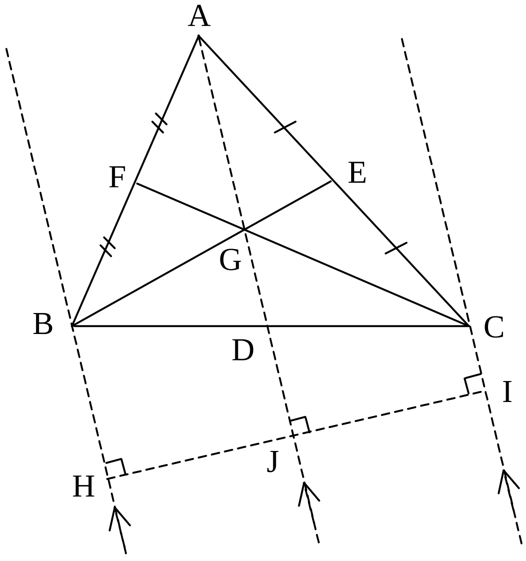


Key Procedures:

1. Construct medians *AD* and *CF*.
2. Construct mid points *H* and *J* of *AG* and *CG* respectively and form quadrilateral *DFHJ*.
3. Consider the Δ*BFD*, Δ*BAC* and then Δ*GJH* and Δ*GCA*, apply the Mid-point Theorem to show that *JH* = *DF* and *JH* // *DF* and hence deduce that *DFHJ* is a parallelogram.
4. Use the properties of parallelogram to find the ratios *FG* : *GJ*, *CG* : *GF* and *AG* : *GD*.

(2) Prove that the medians are concurrent.

Prerequisite: Area of triangle, centroid of a triangle divides the medians in the ratio 2:1, ratio of areas of two triangles having the same height = ratio of their base lengths.



Key Procedures:

1. Construct medians *BE* and *CF*.
2. Construct line *AD*, which passes through the point of intersection of *BE* and *CF*.
3. Construct the lines *BH* and *CI* such that *AD*//*BH*//*CI*.
4. Construct line *HI* such that $HI \perp AD$.
5. Show that the area of Δ*AFG* = the area of Δ*BFG*.
6. Show that the area of Δ*AEG* = the area of Δ*CEG*.
7. Apply the fact *BG* : *GE* = *CG* : *GF* = 2 : 1 to show that the area of Δ*AFG* = the area of Δ*AEG*.
8. Apply the result of 7 to show that *HJ* = *JI*.
9. Show that the area of Δ*BGD* = the area of Δ*CGD*.
10. Find the ratio *BD* : *DC* and hence show that *AD* is the median of the triangle.

Reference：

Books and Articles:

1. Battista, M. T. (1998). *Shape makers: developing geometric reasoning with the Geometer's Sketchpad*. Emeryville, California: Key Curriculum Press.
2. Dixon, Robert A. (1991). *Mathographics*. New York: Dover Publications.
3. Perham, A. E., Perham, B. H., Perham, F. L. (1997). Creating a Learning Environment for Geometric Reasoning. In *Mathematics Teachers,* 90(7), pp.521 – 524. Reston, Virginia: National Council of Teachers of Mathematics.
4. Wyatt, K. W., Lawrence, A. and F., Gina M. (1998). *Geometric activities for middle school students: with the Geometer's Sketchpad*. Emeryville, California: Key Curriculum Press.
5. Yerushalmy, M. and Houde, R. (1987). *Geometry problems and projects: triangles*. Pleasantville, New York: Sunburst Communications.
6. 中國教育學會主辦。《中小學數學—初中版》。1999 年第 7 – 8 期 (頁八)。

Web Sites:

1. http://www.geom.umn.edu/~demo5337/Group2/trianglecenters.html.
2. http://cedar.evansville.edu/~ck6/tcenters/index.html.
3. http://mathworld.wolfram.com/MedianTriangle.html.