Exemplar 4 To Study the Effects of Translation, Reflection and Rotation on Points in a Coordinate Plane

Objectives：

(1) To write down the coordinates of the image of a point under translation, reflection with respect to lines parallel to the *x* axis and the *y*-axis and rotation about the origin through multiples of 90° on points in a coordinate plane

(2) To use the symbol $(x, y) \rightarrow (x', y')$ to describe the transformation

Dimension： Measures, Shape and Space

Learning Unit： Introduction to Coordinates

Key Stage： 3

Materials Required： *Geometer's Sketchpad*

Prerequisite Knowledge： Coordinates, intuitive idea in translation, reflection and rotation

Description of the Activity：

1. Students are grouped in pairs for the activity.
2. Students are asked to
   (a) construct a quadrilateral ABCD in a rectangular coordinate plane (an example is shown in Figure 4.1),
   (b) measure the coordinates of the vertices,
   (c) plot four points P(2,6), Q(5,6), R(5,4) and S(1,0),
   (d) select any two points from P, Q, R and S and mark a vector joining them,
   (e) translate the quadrilateral ABCD by the vector in (d),
   (f) label the image as A'B'C'D' such that A→A', B→B', C→C' and D→D' as shown in Figure 4.1,
   (g) complete Part I in Worksheet 4.1.

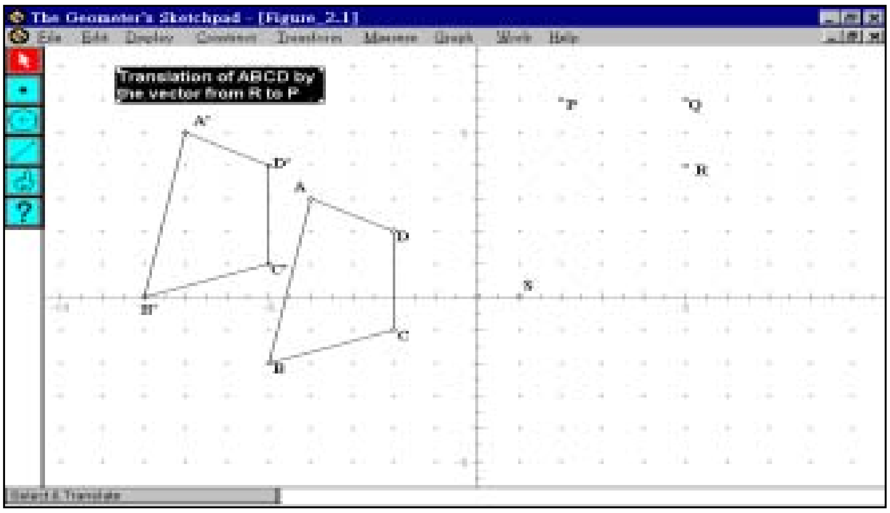

Figure 4.1

3. Some students are invited to present their findings to the class. The teacher can make comment when appropriate.

4. Students are asked to clear the screen for a new sketch. Then they are required to
   (a) construct a quadrilateral ABCD in the rectangular coordinate plane (an example is shown in Figure 4.2),
   (b) measure the coordinates of the vertices,
   (c) reflect the quadrilateral ABCD (i) about the *x*-axis,
   (ii) about a line parallel to the *x*-axis,
   (iii) about the *y*-axis,
   (iv) about a line parallel to the *y*-axis,
   (d) label the image as A'B'C'D' such that A→A', B→B', C→C' and D→D' as shown in Figure 4.2 for the case (c)(ii),
   (e) complete Part II in Worksheet 4.1.

5. Some group representatives are invited to present their findings to the class. The teacher can make comments.

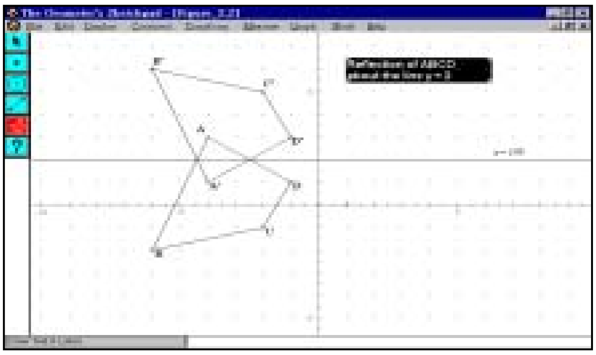

Figure 4.2

6. Students are asked to clear the screen for another new sketch. Then they are told to
   (a) construct a pentagon ABCDE in the rectangular coordinate plane,
   (b) measure the coordinates of the vertices,
   (c) rotate the pentagon ABCDE about the origin through an angle as specified in Table 4.3 in Part III of Worksheet 4.1,
   (d) label the image as A'B'C'D'E' such that A→A', B→B', C→C', D→D' and E→E' as shown in Figure 4.3,
   (e) answer Questions 1 to 3 in Part III of Worksheet 4.1.

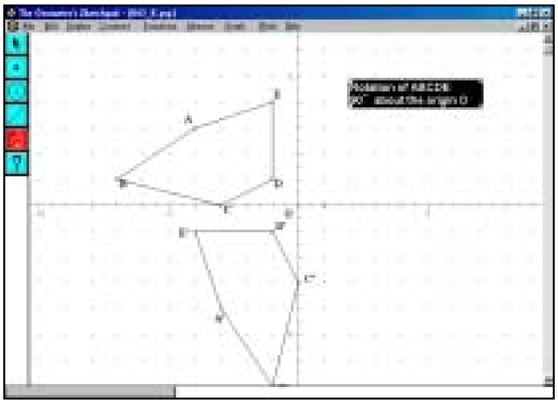

Figure 4.3

7. The teacher asks students to do the following investigation and answer Question 4 in Part III of Worksheet 4.1.
   (a) When the angle of rotation about the origin is a negative multiple of 90°, what is the effect on the direction of turn?
   (b) Use symbols to represent each rotation performed in (a).

8. Choose any vertex of the pentagon ABCDE as the centre of rotation. Use symbols to represent the rotations through multiples of 90° about this point.

Angle of rotation	Symbols representing the rotation
90°	$(x, y) \to (\quad , \quad)$
180°	$(x, y) \to (\quad , \quad)$
270°	$(x, y) \to (\quad , \quad)$
360°	$(x, y) \to (\quad , \quad)$
-90°	$(x, y) \to (\quad , \quad)$
-180°	$(x, y) \to (\quad , \quad)$
-270°	$(x, y) \to (\quad , \quad)$
-360°	$(x, y) \to (\quad , \quad)$

Worksheet 4.1: Translation, Reflection and Rotation of Points in the Coordinate Plane

Part I：Translation

Instruction：

1. Construct a quadrilateral ABCD in a rectangular coordinate plane.
2. Measure the coordinates of the vertices.
3. Plot four points P(2,6), Q(5,6), R(5,4) and S(1,0).
4. Select any two points from P, Q, R and S. Mark a vector joining them.
5. Translate the quadrilateral ABCD by the vector in Point 4.
6. Label the image as A'B'C'D' such that A→A', B→B', C→C' and D→D'.

Investigation：

1. Complete Table 4.1.

Vector		A(,)	B(,)	C(,)	D(,)
From	To				
P	Q	A'(,)	B'(,)	C'(,)	D'(,)
Q	P	A'(,)	B'(,)	C'(,)	D'(,)
R	Q	A'(,)	B'(,)	C'(,)	D'(,)
Q	R	A'(,)	B'(,)	C'(,)	D'(,)
R	P	A'(,)	B'(,)	C'(,)	D'(,)
Q	S	A'(,)	B'(,)	C'(,)	D'(,)

Table 4.1

2. Do you see a relationship between the coordinates of the vertices of the original quadrilateral and those of the image? If there is, briefly describe the relationship.

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3. Drag the vertices of the quadrilateral ABCD to different points on the grid. Does the relationship still hold? Write your conjecture in the space below.

Conjecture:

Part II：Reflection

Instruction：

1. Construct a quadrilateral ABCD in the rectangular coordinate plane.

2. Measure the coordinates of the vertices.

3. Reflect quadrilateral ABCD
   (a) about the x-axis,
   (b) about a line parallel to the x-axis: (i) y = -1, (ii) y = 2,
   (c) about the y-axis,
   (d) about a line parallel to the y-axis: (i) x = -1, (ii) x = 2,

4. Label the image as A'B'C'D' such that A→A', B→B', C→C' and D→D'.

Investigation：

1. Complete Table 4.2.

Axis of reflection (Mirror line)	**A(,)**	**B(,)**	**C(,)**	**D(,)**
(a) x-axis	A'(,)	B'(,)	C'(,)	D'(,)
(b) y-axis	A'(,)	B'(,)	C'(,)	D'(,)
(c) a line parallel to x-axis				
(i) y = -1	A'(,)	B'(,)	C'(,)	D'(,)
(ii) y = 2	A'(,)	B'(,)	C'(,)	D'(,)
(d) a line parallel to y-axis				
(i) x = -1	A'(,)	B'(,)	C'(,)	D'(,)
(ii) x = 2	A'(,)	B'(,)	C'(,)	D'(,)

Table 4.2

2. Do you see a relationship between the coordinates of the vertices of the original quadrilateral and those of the image? If there is any, briefly describe the relationship.

3. Drag the vertices of quadrilateral ABCD to different points on the grid. Does the relationship still hold? Write your conjecture in the space below.

Conjecture:

4. Reflect the quadrilateral ABCD about the y-axis and then follow by a reflection about the x-axis. How is the image related to the original quadrilateral?

Part III：Rotation

Instruction：

1 Construct a pentagon ABCDE in the rectangular coordinate plane.

2 Measure the coordinates of the vertices.

3 Rotate the pentagon ABCDE about the origin through an angle as specified in Table 4.3.

4 Label the image as A'B'C'D'E' such that A→A', B→B', C→C', D→D' and E→E'.

Investigation：

1. Complete Table 4.3.

Angle of rotation	Direction of turn*	A(,)	B(,)	C(,)	D(,)	E(,)
90°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
180°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
270°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
360°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)

Note：* Clockwise / Anti-clockwise

Table 4.3

2. Do you see a relationship between the coordinates of the vertices of the original quadrilateral and those of the image? If there is any, briefly describe the relationship.

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3. Drag the vertices of ABCDE to different points on the grid. Does the relationship still hold? Write down your conjecture for the rotations in the case of 90°, 180°, 270° and 360°.

Conjecture:

4. (a) When the angle of rotation about the origin is a **negative** multiple of 90°, guess what the direction of turn is.

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(b) Complete Table 4.4.

Angle of rotation	Direction of turn*	A(,)	B(,)	C(,)	D(,)	E(,)
-90°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
-180°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
-270°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)
-360°		A'(,)	B'(,)	C'(,)	D'(,)	E'(,)

Note：* Clockwise / Anti-clockwise

Table 4.4

(c) Use symbols to represent the rotations of the points (x,y) in the case of 90°, 180°, 270°, 360°, -90°, -180°, -270° and -360°.

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5. **Challenging Question:**

   Express the rotation of the vertex, (x,y), a figure about another vertex, (a,b), through 90^o in symbols.

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Notes for Teachers：

1. The teacher should explain to students what a vector is when using the software to do the activity concerning translation. There is no need to teach them in depth about vectors. A vector is just a directed line segment. It indicates a direction of the move and its length represents how far the move is when performing translation.
2. The teacher should introduce how to use symbols representing the transformation before the activities.
3. Suggested answers for Worksheet 4.1：

Part I：Translation

Vector		Symbols representing the translation
From	**To**	
P	Q	$(x, y) \rightarrow (x+3, y)$
Q	P	$(x, y) \rightarrow (x-3, y)$
R	Q	$(x, y) \rightarrow (x, y+2)$
Q	R	$(x, y) \rightarrow (x, y-2)$
R	P	$(x, y) \rightarrow (x-3, y+2)$
Q	S	$(x, y) \rightarrow (x-4, y-6)$

Part II：Reflection

Axis of reflection (Mirror)	Symbols representing the translation
(a) x-axis	$(x, y) \rightarrow (x, -y)$
(b) y-axis	$(x, y) \rightarrow (-x, y)$
(c) a line // x-axis (i) y = -1 (ii) y = 2	 (i) $(x, y) \rightarrow (x, -y-2)$ (ii) $(x, y) \rightarrow (x, -y+4)$
(d) a line // y-axis (i) x = -1 (ii) x = 2	 (i) $(x, y) \rightarrow (-x-2, y)$ (ii) $(x, y) \rightarrow (-x+4, y)$

A reflection about the y-axis followed by a reflection about the x-axis is represented by $(x, y) \rightarrow (-x, -y)$. It is equivalent to a clockwise or anti-clockwise rotation of 180° about the origin. The teacher should guide students to discover these equivalent transformations after the completion of Part III Rotation.

Part III：Rotation

Angle of rotation	Symbols representing the rotation
90°	$(x, y) \rightarrow (-y, x)$
180°	$(x, y) \rightarrow (-x, -y)$
270°	$(x, y) \rightarrow (y, -x)$
360°	$(x, y) \rightarrow (x, y)$

Conjecture for point 3:

A positive angle of rotation gives a turn in anti-clockwise direction while a negative angle of rotation gives a turn in clockwise direction.

Answer for point 4:

When the angle of rotation is negative, the direction of rotation is anti-clockwise. (An anti-clockwise rotation of an angle of θ about the origin is equivalent to a clockwise rotation of an angle 360°－θ.)

Angle of rotation	Symbols representing the rotation
-90°	$(x, y) \rightarrow (y, -x)$
-180°	$(x, y) \rightarrow (-x, -y)$
-270°	$(x, y) \rightarrow (-y, x)$
-360°	$(x, y) \rightarrow (x, y)$

4. Answer for the Challenging Question:
   The rotation of the vertex of a figure, (x,y), about another vertex, (a,b), through 90^o can be represented by $(x,y) \rightarrow (-y + a + b, x - a + b)$.

Operation Procedure：

(I) Construct the rectangular coordinate plane

1. Click **Graph** tool in the toolbar. Select **Create Axes**.
2. Click **Graph** tool. Select **Show Grid**.

(II) Construct a polygon

1. Click the **Segment** tool to construct a polygon in the rectangular coordinate plane.
2. Click the **Text** tool to label each vertex.
3. Double click the label if renaming of the vertex is required.

(III) Measure the coordinates of points

1. Click the **Selection Arrow** tool, hold down the **Shift** key and then select all the points.
2. Click the **Measure** tool. Then select **Coordinates**.

(IV) Plot and label points

1. Click the **Graph** tool. Select **Plot Points**.
2. Enter the coordinates of the point(s) to plot. Click **OK**.
3. Click the **Text** tool to label the points plotted, say P, Q, R and S.

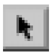
(V) Translate a figure

1. Click the **Selection Arrow** tool, hold down the **Shift** key and then select any two plotted points, say P and R.
2. Click the **Transform** tool. Select **Mark Vector**.
3. Click to select the figure to translate. (Hold down the **Shift** key if more than one object is selected.)
4. Click the **Transform** tool. Select **Translate**.
5. Click the **Text** tool to label the image.

(VI) Reflect a figure

1. Click the **Selection Arrow** tool  . Click to select the axis of reflection (mirror line), say the *x*-axis. (If the axis of reflection is a line other than the axes, then use the **Line** tool to draw the axis before this step.)
2. Click the **Transform** tool. Select **Mark Mirror**.
3. Click to select the figure to reflect. (Hold down the **Shift** key if more than one object is selected.)
4. Click **Transform** tool. Select **Reflect**.
5. Click the **Text** tool  to label the image.

(VII) Rotate a figure

1. Click the **Selection Arrow** tool  . Select a point in the plane as the centre of rotation. (If the rotation is performed about the origin, then the centre of rotation is the origin.)
2. Click **Transform** tool. Select **Mark Centre**.
3. Click to select the figure to rotate. (Hold down the **Shift** key if more than one object is selected.)
4. Click **Transform** tool. Select **Rotate**.
5. Enter an angle of rotation, say 90º, and then click **OK**.
6. Click the **Text** tool  to label the image.