Exemplar 

Trigonometric Identities

Objectives： To explore the trigonometric identities：

(1) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(2) $\sin^2\theta + \cos^2\theta = 1$

(3) $\sin(90° - \theta) = \cos\theta$

(4) $\cos(90° - \theta) = \sin\theta$

(5) $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$

Dimension： Measures, Shape and Space

Learning Unit： Trigonometric Ratios and Using Trigonometry

Key Stage： 3

Materials Required： *Excel* and the file 09_ex_e.xls

Prerequisite Knowledge： Definitions of the trigonometric ratios for acute angles

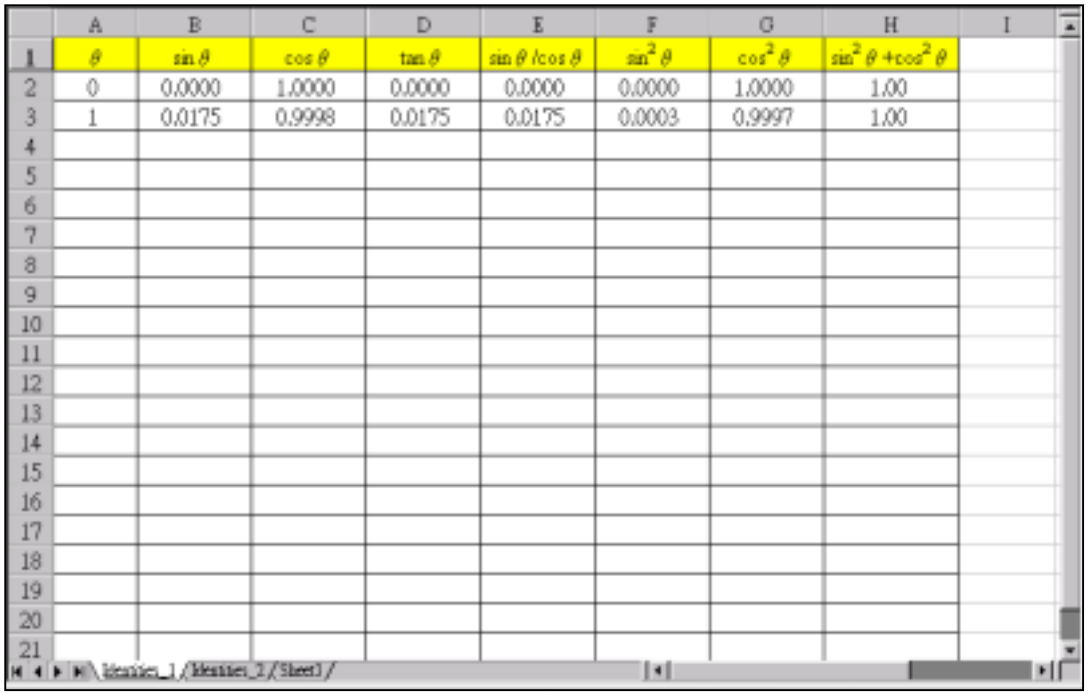
Description of the Activity：

1. The teacher reviews the definitions of sine, cosine and tangent of an acute angle at the beginning of the lesson.

2. The teacher divides students into groups and distributes Worksheet 9.1 and the file 09_ex_e.xls in a diskette to students.

3. Students are asked to complete Part I of Worksheet 9.1 by using the worksheet "Identities_1" in the *Excel* file "09_ex_e.xls" provided in the diskettes. See Figure 9.1. The purpose of the worksheet "Identities_1" is to explore the trigonometric identities

   $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.



	A	B	C	D	E	F	G	H	I
1	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sin\theta/\cos\theta$	$\sin^2\theta$	$\cos^2\theta$	$\sin^2\theta+\cos^2\theta$	
2	0	0.0000	1.0000	0.0000	0.0000	0.0000	1.0000	1.00	
3	1	0.0175	0.9998	0.0175	0.0175	0.0003	0.9997	1.00	
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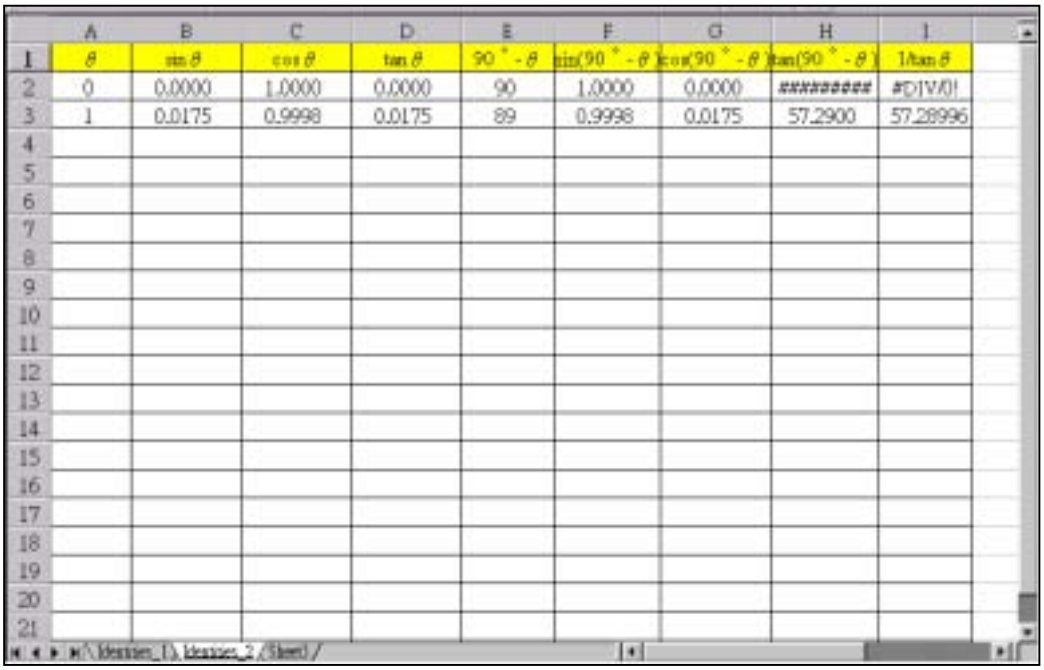
Figure 9.1

4. Some group representatives are asked to present their conjectures and proofs. The teacher can make comments at appropriate times.

5. Students are asked to use the worksheet "Identities_2" in the same file as shown in Figure 9.2 to explore other trigonometric relations. Complete Part II of Worksheet 9.1. The purpose of the worksheet "Identities_2" is to explore the trigonometric identities $\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$ and $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$.



	A	B	C	D	E	F	G	H	I
1	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$90^\circ - \theta$	$\sin(90^\circ - \theta)$	$\cos(90^\circ - \theta)$	$\tan(90^\circ - \theta)$	$1/\tan\theta$
2	0	0.0000	1.0000	0.0000	90	1.0000	0.0000	##########	#DIV/0!
3	1	0.0175	0.9998	0.0175	89	0.9998	0.0175	57.2900	57.28996
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Figure 9.2

6. Some group representatives are asked to present their conjectures and proofs. The teacher can make comments at appropriate times.

Worksheet 9.1: To Explore Trigonometric Identities

Part I To explore the relations among sinθ, cosθ and tanθ.

1. Open the *Excel* file "09_ex_e.xls" and select worksheet "Identities_1".

2. Input the values of θ from 2° to 90° in cells A4 to A92.

3. Calculate the corresponding values of sinθ, cosθ, tanθ, sinθ/cosθ, $\sin^2\theta$, $\cos^2\theta$ and $\sin^2\theta + \cos^2\theta$.

4. By referring to the values obtained in step 3, answer the following questions:
   (a) As θ increases from 0° to 90°, the value of sinθ increases / decreases* from ____ to ____.
   (b) As θ increases from 0° to 90°, the value of cosθ increases / decreases* from ____ to ____.
   (c) As θ increases from 0° to 90°, the value of tanθ increases / decreases* from ____ to ____.

   * *delete the inappropriate word*

5. Write down conjectures.
   (a) What is the relation among sinθ, cosθ and tanθ?

   _____

   _____

   _____

   (b) What can you say about the value of $\sin^2\theta + \cos^2\theta$?

   _____

   _____

   _____

6. Enter different non-integral values of θ such as 32.5°, 65.8°, 89.9°, etc.
Repeat the calculation stated in step 3.
Do your conjectures in step 5 still hold?

_____

_____

_____

7. To prove the conjectures:
(a) Express the trigonometric ratios in terms of a, b and c.

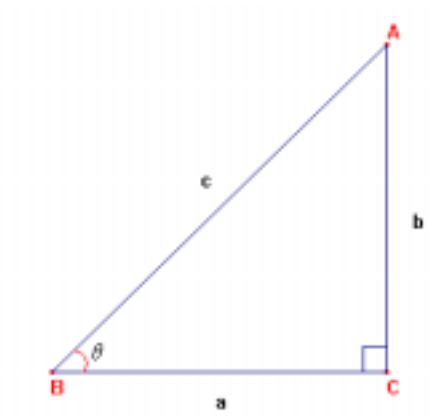


Figure 9.1a

(i) $\sin\theta =$ _____

(ii) $\cos\theta =$ _____ ..... (*)

(iii) $\tan\theta =$ _____

(b) (i) Using the results of (a) (i) and (a) (ii), find $\frac{\sin\theta}{\cos\theta}$ in terms of a, b and c.

_____

_____

_____

_____

_____

(ii) Comparing your result in b(i) with that in (a) (iii), what do you notice? Write down your conclusion.

_____

_____

(c) Using the results of (a) (i) and (a) (ii), prove that of $\sin^2\theta + \cos^2\theta=1$.

Part II To explore the relations among sinθ, cosθ, tanθ, sin(90° – θ), cos(90° – θ) and tan(90° – θ).

1. Open the *Excel* file "09_ex_e.xls" and select worksheet "Identities_2".
2. Input the values of θ from 2° to 90° in cells A4 to A92.
3. Calculate the corresponding values of sinθ, cosθ, tanθ, 90° – θ, sin(90° – θ), cos(90° – θ), tan(90° – θ) and $\frac{1}{\tan\theta}$.
4. Write down all the relations you can find from the results.

_____

_____

_____

_____

_____

5. Enter different non-integral values of θ and repeat the calculations stated in step 3. Check whether the relations in step 4 still hold.

_____

_____

_____

6. To prove the conjectures:

(a) Express the angle A in terms of θ and sinA, cosA, tanA, sin(90° – θ), cos(90° – θ) and tan(90° – θ) in terms of a, b and c.

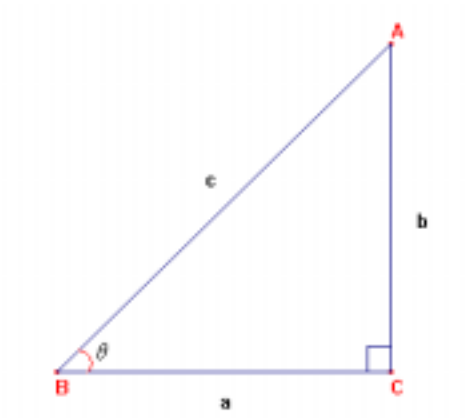


Figure 9.1b

A = _____

sinA = _____

cosA = _____

tanA = _____

sin(90° – θ) = _____

cos(90° – θ) = _____ ..... (**)

tan(90° – θ) = _____

(b) Comparing the results in (**) and those in (*) in step 7 of Part I, what can you conclude?

_____

_____

_____

Notes for Teachers:

1. The teacher needs to prepare the *Excel* file before the lesson.

2. In *Excel*, the calculations of built-in trigonometric functions are in the radian measure. Some conversion must be made to change the input angle from the degree measure to the radian measure when the built-in functions are used in calculations.

3. The teacher should bring to students' attention that, if any one trigonometric ratio of an angle is given, the two identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$ may be used to calculate the value of any other ratios of that angle. Supplementary exercises should be given to students to consolidate the concept.

4. In Figure 9.1b, $\angle A$ and $\angle B$ are called complementary angles. The teacher needs to introduce this term to students.

5. For more able student, the teacher can delete the column "1/tanθ" in Figure 9.2.

6. When using Microsoft *Excel*, students will obtain the notation "#########" as they evaluate the values of tanθ for θ = 90° and tan (90° - θ) for θ = 0°. The teacher needs to clarify its meaning with students. In the software, this notation represents an infinite number while in mathematical sense, we say that tan90° is undefined.

7. The teacher can use other software such as *Graphmatica* to let students understand the trigonometric identities visually.
   For example: In order to investigate the identity cos(90° – θ) = sinθ graphically, the teacher can use the software *Graphmatica* to draw the graphs of y = sin x and y = cos (90° – x) for 0° ≤ x ≤ 90°. Students will find that the two graphs coincide with each other.

8. Suggested answers to worksheet 9.1:
   Part I
   Point 4 (a) As θ increases from 0° to 90°, the value of sinθ increases from 0 to 1.
   (b) As θ increases from 0° to 90°, the value of cosθ decreases from 1 to 0.
   (c) As θ increases from 0° to 90°, the value of tanθ increases from 0 to infinity.
   Point 5 (a) The relation is $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
   (b) The relation is $\sin^2\theta + \cos^2\theta = 1$.

Point 7 (a) (i) $\sin\theta = \frac{b}{c}$ (ii) $\cos\theta = \frac{a}{c}$ (iii) $\tan\theta = \frac{b}{a}$.

(b) (i) $\frac{\sin\theta}{\cos\theta} = \frac{b}{a}$ (ii) $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

Part II

Point 4 $\sin(90°-\theta) = \cos\theta$, $\cos(90°-\theta) = \sin\theta$, $\tan(90°-\theta) = \frac{1}{\tan\theta}$.

Point 6 (a) $A=90°-\theta$.

$\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$ and $\tan A = \frac{a}{b}$.

$\sin(90°-\theta) = \frac{a}{c}$, $\cos(90°-\theta) = \frac{b}{c}$ and $\tan(90°-\theta) = \frac{a}{b}$.

(b) $\sin(90°-\theta) = \cos\theta$, $\cos(90°-\theta) = \sin\theta$, $\tan(90°-\theta) = \frac{1}{\tan\theta}$.

Operation Procedure :

(I) Copy the symbol θ from a Word document into the cell A1:

1. Open a new Word document.
2. Select **Insert** tool from the **Menu** toolbar and choose **Symbol**.
3. Select **Normal Text** in the **Font** box and **Basic Greek** in the **Subset** box.
4. Click the symbol θ, press **Insert** and then press **Close**.
5. Copy the symbol θ.
6. Open a new *Excel* file. Click the cell A1 and then paste the symbol θ.
7. (In English Window NT 4.0 only) Change the font to Arial to display the symbol θ.

(II) Enter the term $\sin^2\theta$ in the cell F1:

1. Select the cell F1 and then enter sin2θ.
2. Highlight the number '**2**' in sin2θ and then select **Format** in the **Menu** toolbar.
3. Select **Cells** and click **Superscript** in the **Special Effect** box.
4. Press **OK**.

(III) Input values of θ in cell A2 to A92:

1. Enter the value '**0**' in cell A2 to set the initial value of θ at 0.
2. Enter the formula '= A2 + 1' in cell A3 to calculate the value of θ + 1.
3. Select cell A3. Go to the lower-right corner of cell A3. Drag the fill handle to cell A92 in order to copy the formula of A3 to cells A4 to A92.

(IV) Perform calculations in the activity:

1. To calculate the value of sinθ:
   (a) Label cell B1 as sinθ.
   (b) Select cell B2. Enter the formula '= sin(A2*PI()/180)' into cell B2 to calculate the value of sinθ.
   (c) Enter the formulas of cosθ and tanθ in cells C2 and D2 in a similar way.

2. To calculate the values of $\frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta$:

   (a) Label cell E1 as sinθ/cosθ.

   (b) Select cell E2. Enter the formula '= B2 / C2' into cell E2 to calculate the value of $\frac{\sin\theta}{\cos\theta}$.

   Then press **Enter**.

   (c) Label cell F1 as $\sin^2\theta$.

   (d) Select cell F2. Enter the formula '=B2^2' into cell F2 to calculate the value of $\sin^2\theta$. Then press **Enter**.

   (e) Enter the formulas of $\cos^2\theta$, $\sin^2\theta + \cos^2\theta$, $90° - \theta$, $\sin(90° - \theta)$, $\cos(90° - \theta)$, $\tan(90° - \theta)$ and $1/\tan\theta$ in a similar way.

(V) To investigate the trigonometric identities graphically by using *Graphmatica*:

1. An evaluation version of *Graphmatica* can be downloaded at http://www8.pair.com/ksoft/.

2. Select **View** | **Grid Range**.

3. Enter the following data in the **Grid Range** Dialogue Box:

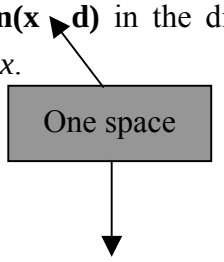
Left : **0**	Right : **90**
Bottom : **-1.5**	Top : **1.5**

   Then press **OK**.

4. Select **Labels** | **Legends**.

5. In the **Axis Legends** Dialogue Box, click **Custom Spacing** and enter **10** in the box of the *x*-axis. Then press **OK**.

6. Enter the equation **y=sin(x d)** in the dialogue box below the icon bar to draw the graph of $y = \sin x$.



   Then press **Ente**r.

   Enter the equation **y=cos((90–x) d)** in the dialogue box to draw the graph of $y = \cos(90° - x)$ and then press **Enter**.

7. Select **View** | **Colors** to change the colours of the curve and background.