Exemplar 6: Exploration of the Formula for the Area of a Circle

Dimension: Measures, Shape and Space

Learning Unit: Simple Idea of Areas and Volumes

Key Stage: 3

Materials Required: *Sketchpad* for Approach 2 in Part A

Prerequisite knowledge: The formula for the area of a triangle and the circumference of circle

Key Features：
This exemplar consists of 2 parts. The first part is to let students explore the formula for the area of a circle in two approaches with different levels of difficulty to cater for the different learning abilities of students. The second part of the exemplar is to ask students to carry out a project work to investigate the methods of estimating the value of π. This part targets for the average and more able students as they are expected to spend less time in part A and are more available in part B.

Part		Less able students	Average students	More able students
A	Approach 1	✓	✓ or	
	Approach 2		✓	✓
B			✓	✓

Remark : ✓ represents the part(s) that can be participated by students when they start to learn the captioned topic.

Part A：To derive the formula for the area of a circle.
Approach 1：This activity provides a simple way to show the formula for the area of a circle. A circle is dissected into even number of very small sectors to form a figure that approximates a parallelogram. From the parallelogram formed, the formula for the area of a circle can be deduced.

Approach 2： This activity is a little bit difficult compared with the activity in approach 1. Students can approximate the area of a circle by increasing the number of sides of a regular polygon inscribed in the circle. Through this activity, students can realize and appreciate the past attempts in approximating the area of a circle and the value of π .

Part B：To investigate the methods of estimating the value of π.

This part is a project work that requests students to investigate the methods of estimating the value of π . Students are required to collect information from Internet, video or reference books by themselves. Students can broaden their views of mathematics and will appreciate the efforts made by the mathematicians.

Description of the Activity：

Part A：To derive the formula for the area of a circle.

Approach 1

1. Show how to find the area of a circle by cutting a circle with radius *r* into 8 equal sectors and rearranging these sectors into a shape resembling a parallelogram as in Figure 1.

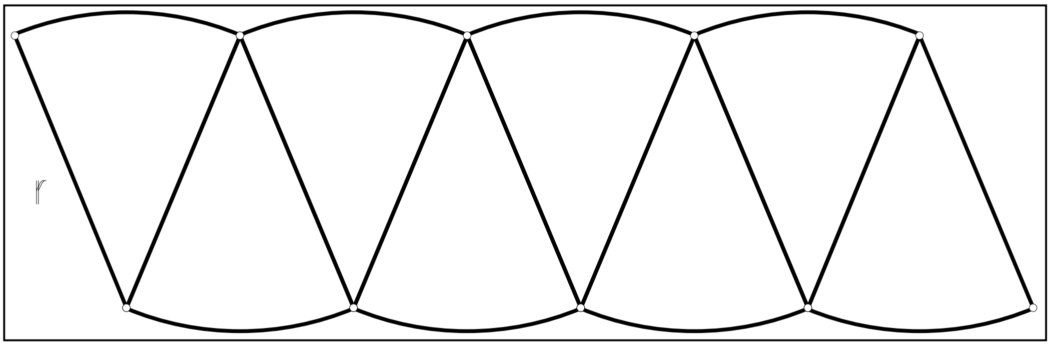

Figure 1

2. Ask students to find the area of this figure. As the “base” of this figure is not a straight line, they can’t find the area by using a simple formula.

3. Then show another circle with the same size as above and divide this circle into 18 equal sectors. Rearrange these sectors as above. Figure 2 is an illustration.

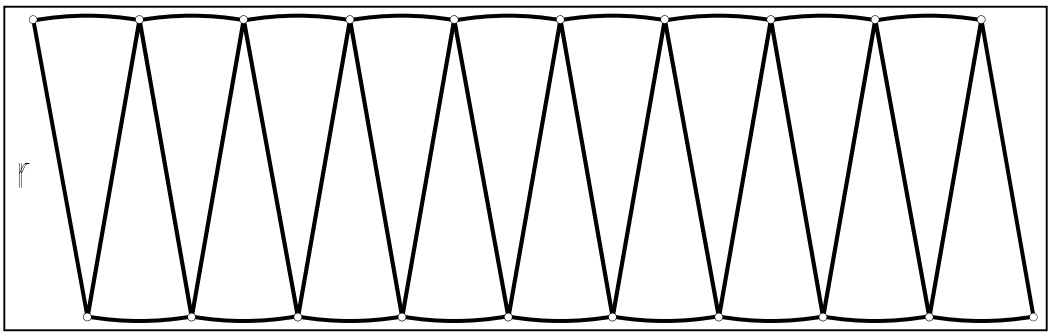

Figure 2

4. Guide students to describe the similarities and differences between these two figures. Then ask students the following questions.
   (a) What is the change of the base of the figure if we make more sectors on the circle?
   (b) What will the figure become if we divide the circle into a large number of sectors?
   (c) What are the length of the height and the base of this figure?
   (d) What is the area of this figure?

5. Ask students to make hypothesis on the formula for the area of a circle and then draw the conclusion on the formula. The teacher should guide students to observe that the sides of the figure are r and πr and with an infinite number of sectors as the limiting case, the figure becomes a rectangle with an area of πr^2. Thus, the area of the circle must be πr^2, too.

Approach 2

1. Show a grid paper to students. On the grid paper there is a circle with radius r. A square EFGH is drawn around the circle so that each side of the square touches the circle. Two perpendicular diameters are drawn so that they divide the circle into four equal parts. They also divide the large square into four equal small squares (Figure 3).

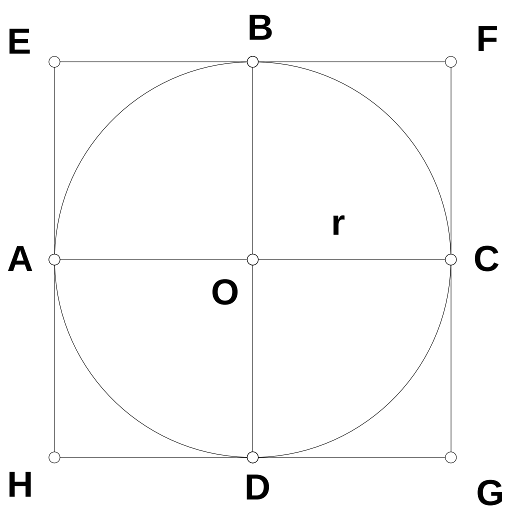

Figure 3

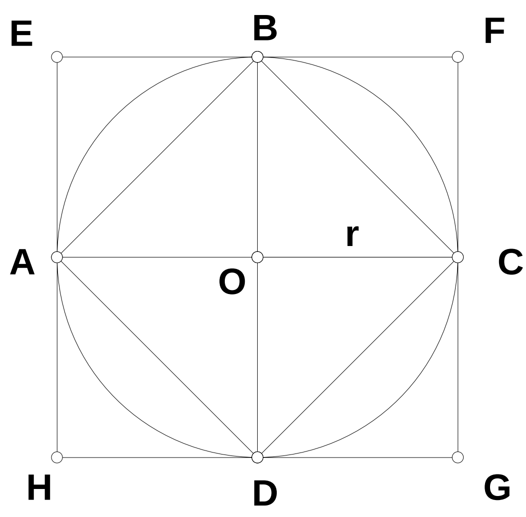

Figure 4

2. Ask the following questions.
   (a) What is the area of the small square OAEB?
   (b) What is the area of the square EFGH?
   (c) Is the circle larger or smaller than square EFGH? Why?

3. Join the end points of the diameters as shown in Figure 4. Ask students the following questions.
   (a) What is the area of the square ABCD?
   (b) Is the circle larger or smaller than square ABCD? Why?
   (c) According to the above results, what is the range of the area of the circle?

4. Give a *Sketchpad* file cirarea.gsp to students and ask them to open it (See Figure 5). This program is to approximate the area of a circle by inscribing regular polygons with varying number of sides inside the circle. Students are asked to drag the point "Drag" along the segment and observe how the polygon and the measures change.

5. Ask students the following questions:
   When you drag the point "Drag" to the right,
   (a) what does the polygon change?
   (b) what are the changes in a, b and r? Increase, decrease or no change?
   (c) how the area of the polygon changes when you increase the number of sides of the polygon?

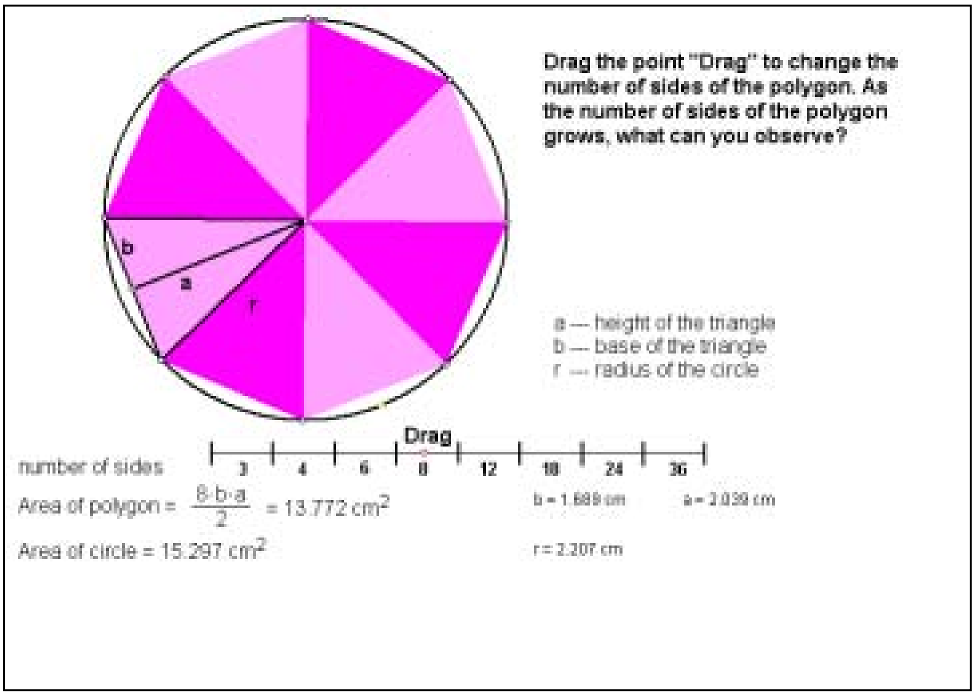

Figure 5

6. Guide students to discover that as the number of sides of the polygon gets larger, the area of the polygon approaches to the area of the circle.

7. Guide students to derive the formula for the area of a circle by referring to Figure 5
Pose the following questions to initiate class discussion.
   (a) What is the area of each triangle? Express it in terms of a and b.
   (b) What is the area of the regular octagon?
   (c) If a regular *n*-sided polygon is inscribed in the circle instead, express the area of the *n*-sided polygon in terms of a, b and *n*.
   (d) As the number of sides of the regular polygon (i.e. the value of *n*) increases, what does a approach?
   (e) What is the perimeter of the regular *n*-sided polygon? What does the perimeter of the polygon approach as *n* increases?
   (f) What is the formula for the circumference of a circle? When *n* tends to infinity, by using the previous results, rewrite the area of the polygon to represent the area of a circle.

8. Introduce the stories of Ancient mathematician including Liu Hui who had used this method to estimate the area of circle and the value of π.

Part B：To investigate the methods of estimating π.

1. Ask students to do a project to investigate the past attempts to estimate the values of π. Some methods include:
   (a) The Archimedes Approach. Approximate the area of a circle by calculating the perimeter of a regular *n*-sided polygon inscribed in the circle and one circumscribed around the circle. The inscribed polygon has a smaller perimeter than the circumference of the circle, and the circumscribed polygon has a larger one. Archimedes used a polygon with 96 sides and proved that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
   (b) Buffon's needle method. Calculate π from the needle drops. Put parallel lines with distance *d* apart on the floor and drop a needle of length *L* onto these lines (with $d > L$). Do this experiment *N* times, and suppose the needle touches one of the lines *K* times. Then π is approximated by $\frac{2LN}{Kd}$, for sufficiently large *N*.
   (c) John Wallis Formula. John Wallis used infinitely many small rectangles to approximate the area of a quarter of a circle. The formula is $\frac{\pi}{2} = \frac{2\times2\times4\times4\times6\times6\times8\times\cdots}{1\times3\times3\times5\times5\times7\times7\times\cdots}$.
2. Students are not expected to know all the mathematical knowledge involved in the methods of estimating π since most of such knowledge is beyond their academic level. Nevertheless, students should understand the processes and the ideas of the methods.

Notes for Teachers：

1. Suggested answers to the questions in Approach 1 of Part A:
   4 (a): The base is like a straight line when we make more sectors on the circle.
   4 (b): The figure will become a parallelogram or a rectangle.
   4 (c): The height of this figure has length r, and the base is half of the circumference of the circle, i.e. πr.
   4 (d): The area of the figure is πr^2.

2. Suggested answers to the questions in Approach 2 of Part A:
   2 (a): r^2.
   2 (b): $4r^2$.
   2 (c): The area of the circle is smaller than the area of square EFGH since the circle is inscribed in the square.
   3 (a): $2r^2$.
   3 (b): The area of the circle is larger than the area of square ABCD since the square is inscribed in the circle.
   3 (c): $2r^2$ < area of the circle with radius r < $4r^2$.
   5 (a): The number of sides of the polygon increases when students drag the point "Drag" to the right.
   5 (b): The values of a, b and r will increase, decrease and be no change respectively.
   5 (c): The area of the polygon will increase and approach to the area of the circle.
   8 (a): $\frac{\mathrm{ab}}{2}$
   8 (b): 4ab.
   8 (c): $\frac{\mathrm{ab}n}{2}$.
   8 (d): The radius of the circle, i.e. r.
   8 (e): The perimeter is bn. When n increases, the perimeter approaches the circumference of the circle.
   8 (f): The formula for the circumference of a circle is $2\pi r$. When n tends to infinity, the area of the polygon tends to $\frac{r \cdot 2\pi r}{2}$, i.e. πr^2.
   Therefore the area of a circle is πr^2.

3. The teacher may consider introducing the history of π in the lesson as a further activity of Activity A. Alternatively, the teacher may request students to do a project about the history of π as in Activity B.

Reference :
Students can obtain useful information in the following:

Web sites:

1. http://forum.swarthmore.edu/dr.math/faq/faq.pi.html

2. The following web sites concerning Archimedes' approximation of π:
   http://www.math.utah.edu/~alfeld/Archimedes/Archimedes.html
   http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

3. For the Buffon's Needle method:
   http://www.angelfire.com/wa/hurben/buff.html
   http://www.mste.uiuc.edu/reese/buffon/buffon.html
   http://www.math.uah.edu/stat/buffon/buffon2.html

4. Others:
   http://gallery.uunet.be/kurtvdb/pi.html
   http://www.geocities.com/hjsmith_geo/download.html#PiW

Books and Articles :

1. Schroeder, L. (1974). Buffon's needle problem: "An exciting application of many mathematical concepts". *Mathematics Teacher*, 67(2), pp. 183 – 186. Reston : National Council of Teachers of Mathematics.

2. Cheney, E.W. and Kincaid, D.R. (1999). *Numerical Mathematics and Computing*. 4th Edition. Pacific Grove, California: Brooks/Cole Publishing Company.

3. Blatner, David (1999). *The Joy of* π. New York: Walker & Co.

Video :
Teacher can find useful information from the following video:
Project Mathematics! ***The Story of Pi*** ©1989 California Institute of Technology, Caltech 1 – 70, Pasadena, CA 91125