Exemplar 3: Constructing Data Set from a given Mean, Median and Mode

Dimension:　　　　　　Data Handling

Learning Unit:　　　　Measures of Central Tendency

Key Stage:　　　　　　3

Materials Required:　　NIL

Prerequisite Knowledge:　(1)　Finding the mean, median and mode from a data set
　　　　　　　　　　　　　　(2)　Solving simple simultaneous linear equations

Key Features

This exemplar is to allow students to construct a data set satisfying the given averages, namely mean, median and mode. This exemplar consists of three activities with gradual increase in the level of difficulty. In each activity, different supports including the instructions given and different levels of questions are provided in guiding students of different learning abilities to solve the given problems. It is summarized as follows.

	Part	Less able students	Average students	More able students
A	Given one average	✓	✓	
B	Given two averages	✓	✓	✓
C	Given three averages	✓	✓	✓

Remark : ✓ represents the part(s) that can be participated by students when they start to learn the captioned topic.

Part A:

It requires students to construct a data set with **a given average**.

For less able students, more structured and guided questions are set to help these students build up the foundation technique in constructing a data set. Students are guided to observe that the data set constructed in some cases may not be unique. For average students, the revision part is omitted. Less guided and more difficult questions are set to motivate students to solve problems. More able students can skip this part.

Part B:

It requires students to construct a set of data with **two given averages**.

This activity is preferably conducted as a game activity. For less able students, they are required to form a data set with a given median and then modify the data set to satisfy the given mean. In addition, average students and more able students have to construct a set of four data with a given median and mode. Students are then asked to construct a data set with the number of data reducing to three but satisfying the same median and mode. Students will then discover that there may be no data that satisfy the given averages. Average students are expected to explain orally with mathematical reasons. For the more able students, they are required to write down the reasons for the case of no solution.

Part C:

It requires students to construct a set of data with **three given averages.**

For the less able students, they are expected to construct a data set that satisfying the three given averages. The problems given to them confined to those that have a solution. The teacher helps them to draw conclusions in coping with the problem. The average students have to solve the same problem and tackle some problems that have no solution. They have to explain orally these conjectures. For the more able students, they have to try several problems that have one solution, no solution or infinitely many solutions and determine whether the solution exists or not. If not, they have to change the number of data (usually increase the number of data) in order to solve the problem. These students are expected to try a lot of different cases by themselves in order to generalize the conditions for only one solution, no solution or infinitely many solutions.

Description of the Activity:

Part A: Constructing a data set with **a given average (such as mean)**

Less able students	Average students	More able students
Revision on mean: 1. Ask students to find the mean of three numbers, say 3, 5 and 10. Construction of data set: 2. Replace the number "10" in the above by x. If the mean equals 6, find the value of x. 3. Find two numbers x and y such that the mean of the numbers 3, y and x is equal to 6. 4. Ask students to compare with other classmates' answers and guide students to observe that there is more than one solution to the problem. 5. Guide students to observe that so long as x, y satisfy $x + y = 15$ then these x, y can be the solution. 6. Ask students to construct three numbers x, y and z with mean equals 6. 7. Ask students to construct another data set that consists of three numbers with mean equals 6. 8. Guide students to observe that so long as x, y and z satisfy $x + y + z = 18$ then these x, y, z can be the solution.	Construction of data set: 1. Find two numbers x and y such that the mean of three numbers 3, y and x is equal to 6. 2. Ask students to compare with other classmates' answers and guide students to observe that there are more than one solution to the problem. 3. Ask students to explain why the solutions are not uniquely determined by relating with the definition of mean and the constraint on $x+y$. 4. Ask students to construct three numbers x, y and z with mean equals 6. 5. Ask students to construct another data set that consists of three numbers with mean equals 6. 6. Discuss with students the "non-uniqueness" of the solutions. 7. Ask students to construct a data set of four numbers with mean equals 10. 8. Discuss the strategy in finding the data set.	

Part A (Cont'd)

Less able students	Average students	More able students
9. Ask students to construct a data set of four numbers with mean equals 10. 10. Compare the strategy in finding the data with that in point 8 above.		

Part B: Constructing a data set with **two given averages**

Less able students	Average students	More able students
1. Divide students into groups. 2. Ask students to construct a set of three numbers with a given median (say 4). 3. Check the answers with their group-mates and ask students to find the mean of their data sets. 4. Write down a mean (say 5) on the blackboard. Ask students to discuss with group-mates to modify their data set to satisfy the new mean without changing the value of the median. 5. Ask some students to present their strategies in modifying the data set. 6. Conclude the strategy used and relate the strategy with that in Part A.	1. Follow instructions 1 to 6 for less able students. 2. Ask students to construct another set of 4 data with another 2 averages given, e.g. median = 4, mode = 5. 3. Discuss with students the strategy to construct the data set. 4. Ask students to construct a set of three data with median = 4 and mode = 5. 5. Discuss on the strategy to construct this data set. 6. Ask students to *explain orally* why the data set cannot be constructed.	1. Follow all instructions 1 to 5 for average students. 2. Choose some students to explain their reasons to the whole class. 3. Ask students to *write down* why the data set cannot be constructed.

Part C: Constructing a data set with **three given averages**

Less able students	Average students	More able students
1. Write down "mean = 3, median = 4 and mode = 4" on the blackboard.	1. Ask students to construct a set of three data with "mean = 3, median = 4 and mode = 4".	1. Divide students into groups and ask students to construct a data set of 3 data with "mean = 3, median = 4 and mode = 4". And then the data set with 4 data, 5 data and so on.
2. Ask students to construct a set of three data satisfying the above conditions.	2. Ask students to construct a set of 4 data (5 data and so on) with the same mean, median and mode as in point 1 above.	
3. Ask the following questions to guide students to discover the strategy in finding the data set.	3. Discuss with students the general strategy in constructing a data set with given mean, median and mode.	2. Discuss with students the strategy to construct the data set.
(a) Among the three averages, what is the first number that you are going to fix for the data set? Why?	4. Ask students if it is possible to find a data set of 3 numbers with mean = 3, median = 4 and mode = 5.	3. Ask students to construct data sets satisfying the conditions below:

(b) Once you fix one of the averages, what constraints will be on the other two numbers? Why?	5. Discuss with students why there is a no solution satisfying the condition.	(a) Construct a set of **three** data with:
		     (i) Mean =5, Median=4, Mode=4
		     (ii) Mean =3, Median=4, Mode=5

(c) Is there any other set of data satisfying the given condition? Why?	6. Raise the following questions for further discussions:	(b) Construct a set of **four** data with:
		     (i) Mean =3, Median=4, Mode=5
		     (ii) Mean =3.5, Median=4, Mode=5
		     (iii) Mean =3, Median=4, Mode=4
		     (iv) Mean =3, Median=4, Mode=3

4. Ask students to construct a set of 4 data (5 data, 6 data and so on) with the same mean, median and mode as in point 1 above.	(a) Is it possible to find a data set with mean = 3, median = 4 and mode = 5? How?	
5. Compare students' answers and ask the same questions as in point 3 above.	(b) Can we increase the number of data in the set to satisfy the same condition as in instruction 2 above?	
6. Discuss with students that the general strategy in constructing a data set with given mean, median and mode.	7. Ask students to present their solutions to the whole class.	

Less able students	Average students	More able students
		4. Ask students to determine whether they can construct data sets satisfying the above conditions. If possible, construct the data sets. If not, modify the number of data used in the data set concerned so that the data can fulfill the requirement. 5. Discuss the answers presented and generalize the conditions under which there is/are <u>no solution</u>, <u>only one solution</u> and <u>infinitely many solutions</u> in constructing a data set of 3 numbers with given 3 averages.

Notes for Teachers :

1. Mean is used in Part A and it is crucial to guide students to find the strategy in constructing the data set satisfying the condition that

    sum of data = mean × number of data.

    This constraint is basically derived from the definition of a mean. The common strategy students used in constructing a data set of 4, say with mean = 3, will be to set all data 3,3,3,3. Students will then find by subtracting one to the first number and then adding one to another number will not change the mean of the data set and so on. Hence, the data set can be 2, 3, 3, 4; 1, 3, 3, 5; 1, 2, 4, 5 ; etc.

    If median is used, it is important to guide students to use the definition of the median. By fixing the middle of the data as the given median, it is free to change the values of any numbers less than the median in front of it and any numbers greater than the median after it. The argument is similar for the case of a given mode.

2. In Part B, mean and median are chosen as the given 2 averages for the less able students. The common strategy is to choose the data set as x_1, 4, x_2 first and then follow the solving strategy in Part A.

    Alternatively, students can construct a data set with the given mean first and then modify the data set to the given median.

    >   e.g. first construct data set as 5, 5, 5, ;   3, 5, 7; etc. and then modify the data set to suit the median (4 in this case)
    >
    >   5, 5, 5 to 3 , 4, 8; 2, 4, 9; 1, 4,10; etc.
    >
    >   3, 5, 7 to 3, 4, 8; 2, 4, 9; etc.

    In addition, the average students and more able students have to construct a data set of four numbers from a given median and mode. Usually students may probably choose the strategy of constructing the data set as

    > x_1, x_2, 5, 5 $(x_1, x_2 \le 5)$;   or   5, 5, x_1, x_2 $(x_1, x_2 \ge 5)$.

    If median is 4, then x_1, x_2, 5, 5 will become x_1, 3, 5, 5 (with $x_1 \le 3$) and the other cases will be disregarded.

    If we reduce the number of data set to three but with the same given conditions, students may discover that there is no solution in this case. Average students need to explain the reasons orally to the whole class while more able students have to use mathematics language to write down their reasons.

    As there are three numbers required in this case, students can fix the median first. The data set becomes x_1, 4, x_2 with $x_1 \le 4 \le x_2$. On the other hand, as the mode = 5, x_1 and x_2 should be equal to 5. This reduces to a contradiction. If students fix the mode first, the data set will become x_1, 5, 5 or 5, 5, x_1. It is impossible that the median equals to 4.

    The teacher is suggested to discuss and compare the two strategies in the above with students.

3. In Part C, the teacher should be careful to choose the values of the three averages so that there must be a solution for the activities for less able students. Some questions that have no solution can be set for more able students.

4. The answers for instruction 3 of Part C of more able students are as follows:
   (a) (i) has solution: 4, 4, 7.
       (ii) no solution. (4 data: -1, 3, 5, 5)
   (b) (i) has solution: -1, 3, 5, 5.
       (ii) has solution:1, 3, 5, 5.
       (iii) infinitely many solutions: x_1, 4, 4, x_2 (where $x_1 \leq 4 \leq x_2$ and $x_1 + x_2 = 4$).
       (iv) no solution. (5 data: no solution; 6 data: one of the solutions is -6, 3, 3, 5, 6, 7)

5. The answers for instruction 5 of Part C for more able students are as follows:

   The conditions in constructing a data set with three or four numbers are:

   Let the given mean, median and mode be a, b and c respectively.

Mean	Median	Mode
a	b	c

   To construct a data set with <u>three</u> numbers.

       If $b \neq c$, there is no solution.

       If $b = c$ and $a \leq c$, the three data are $3a - 2c$, c, c in ascending order.

       If $b = c$ and $a > c$, the three data are c, c, $3a - 2c$ in ascending order.

6. To construct a data set with <u>four</u> numbers.

   If $a \leq b \leq c$, the four data are $4a - 2b - c$, $2b - c$, c, c in ascending order.

   If $a \geq b \geq c$, the four data are c, c, $2b - c$, $4a - 2b - c$ in ascending order.

   If $b = c$, there are infinitely many solutions. The four data are $2a - c - \Delta$, c, c, $2a - c + \Delta$ in ascending order where $\Delta \geq |2(c - a)|$.

   In other cases, there are no solutions.

7. The teacher can use the above results to construct the data set easily. For example, the teacher can set the mean, median and mode as 3, 4 and 3 respectively. Students are asked to construct a data set with four numbers. In this case $a = 3$, $b = 4$ and $c = 3$. As $b \neq c$, by point 6 above there is no solution. In Part C, the more able students need to solve this problem and present to the whole class that there is inconsistency found in constructing the data set. They are expected to use five numbers to construct the data set.

8. Both the teacher and students can use *Excel* to check whether they get the correct data set easily.