Exemplar 5: Exploration of the 4 Basic Operations on Directed Numbers

Dimension : Number and Algebra

Learning Unit : Directed Numbers and the Number Line

Key Stage : 3

Materials Required : Refer to the Description of the Activity

Prerequisite Knowledge : Ideas of ordering on number line, concepts of directed numbers and the four fundamental basic operations involving positive integers

Key Features :

This exemplar consists of three parts that cater for different learning abilities of students as shown in the following table. The three parts are under the same learning topic in the Foundation Part of the Syllabus.

Part		**Activities**	**Less able students**	**Average students**	**More able students**
A	Addition & Subtraction	Magnetic Button Model			✓
		Balloon Model	✓	✓	Optional
	Multiplication	Pattern-finding	✓	✓	✓
		Case studies	✓	✓	✓
		Proofs			✓
	Division	Thermometer Model	✓	✓	✓
B		Card game	✓	✓	✓
C		Extension of card game			✓

Remark : ✓ represents the part(s) that can be participated by students when they start to learn the captioned topic.

Part A : To understand the four fundamental basic operations on directed numbers through daily life examples

(I) Addition and Subtraction

Two different approaches to introduce addition and subtraction of directed numbers, namely, the activity "Magnetic button" model and the "Balloon model" are included in this part. The latter model, which is less demanding, helps students to understand the operations of addition and subtraction of directed numbers on the number line.

(II) Multiplication

There are three activities in this part. The objective of Activity 1 is to help students to get an intuitive idea of the rules in multiplying directed numbers through observing patterns. Students can consolidate those rules through various case studies in Activity 2. Activity 3 is designed for able students only. Students are required to give a formal proof that “the product of two directed numbers of the same sign is positive and that of different signs is negative”.

(III) Division

This part is a case study that allows students to derive the rules of the division of directed numbers.

Part B : A card game

This activity is designed for students to apply the operation procedures and rules learnt in Part A through a game activity. It requests students to use the arithmetic symbols "+", "–", "×", "÷" or "( )" and the four integral directed numbers drawn from a pile of cards with directed numbers ranging from -10 to -1 printed on one side of the cards to form a numerical expression equal to 24. As this part of the activity is not difficult in comparing to the activities in Part A, the activity can be carried out as a contest for all students if time is allowed.

Part C : An extension of the card game in Part B.

Students are asked to count the total number of ways to formulate a numerical expression for four given integral directed numbers. The level of difficulty increases greatly in this part and is targeted for more able students.

Preparation for the game in Part B before the lesson:

The teacher needs to prepare several piles of cards such that

1. each pile consists of 40 cards;
2. one of the numbers -10, -9, -8, … , -3, -2, -1 is printed on one side of the card.

-10	-9	-8	-7	-6
-5	-4	-3	-2	-1

3. Each number appears four times in the pile.

Description of the Activity :

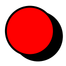
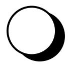
Part A :

(I) Addition and Subtraction

Approach 1 : “Magnetic-button” model

Materials Required : Magnetic buttons, white board.

Details of the Activity :

1. Prepare 10 red magnetic-buttons and 10 white magnetic-buttons.
2. Let students decide which colour represents positive and which represents negative. In the following description, we suppose that the red colour represents positive.
3. Each button represents one unit. Therefore,  represents “＋1” while  represents “-1”.
4. Explain the idea of addition and subtraction of directed numbers using the magnetic-buttons. These include:
   (a) one red button and one white button make a “zero”. That is, “＋1” plus “-1” equals 0;
   (b) **addition** of directed numbers means the **adding of** corresponding numbers of red or white buttons depending on the sign of the number;
   (c) **subtraction** of directed numbers means **taking away** of corresponding numbers of red or white buttons depending on the sign of the number.

5. Use the examples suggested in Table 1 or any other additional examples in guiding students to observe the general pattern. The following questions can be raised during the illustration:

Addition

i) How many buttons are left? What colour are they?

ii) Which directed number do these buttons left represent?

iii) For 2 numbers of the same sign, will addition change the sign of the resulted numbers?

iv) For 2 numbers of opposite signs in cases 3 and 7, how many "zero"s are produced? What is the sign of the resulted number? Which number in the original 2 numbers will affect the sign of the resulted number? What is the general feature of this number ("-4" in case 3 or "+4" in case 7) in comparing to another number?

Subtraction

i) We only have two red buttons but we need to take away four red ones, what should we do?

ii) How many "zero"s should we add in each case? Explain.

iii) How many buttons are left after the required number of buttons are taken away? What colour are they?

iv) Which directed number do these buttons left represent?

v) For 2 numbers of the same sign, will the resulted number take the opposite sign of the original numbers after subtraction? Does it always change in sign after subtraction (by referring to cases 4 and 8)?

vi) For 2 numbers of opposite signs, which number of the original numbers will affect the sign of the resulted number after subtraction?

vii) What is the general feature of this dominated number ("+4" in case 4 or "-4" in case 8) in comparing to another number?

6. Guide students to observe that

i) taking away red buttons is equivalent to adding the same amount of white buttons i.e. $a-(+b)=a+(-b)$ in mathematical sense;

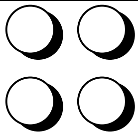
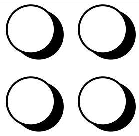
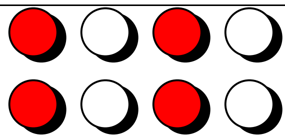
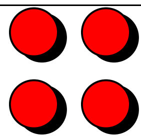
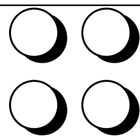
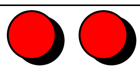
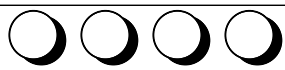
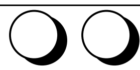
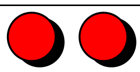
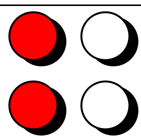
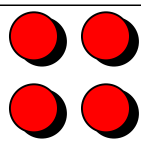
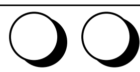
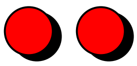
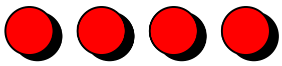
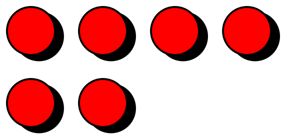
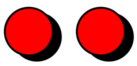
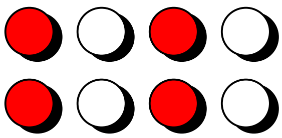
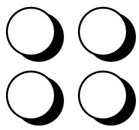
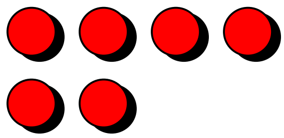
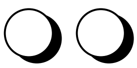
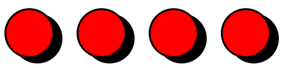
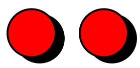
ii) the resulted number of adding 2 numbers of the same sign will have the same sign of the numbers but this is not true for the case of subtraction;

iii) the resulted number of adding 2 numbers of different signs will have the same sign of the number which has a greater magnitude (dominated number) and the magnitude of the resulted number will be the same as the difference of the magnitudes of the 2 numbers as in case 4.

ADDITION AND SUBTRACTION

When dealing with addition, pair the red and white buttons to make zero(s) and count what is left. When dealing with subtraction, if there are sufficient number of buttons, just take away them. Otherwise, add enough "zeros" i.e. pair(s) of red and white buttons, to make the number of buttons to the desired quantity.

Some examples are given in Table 1.

Case	Expression	Start	Buttons added	Buttons taken away	Buttons left
1	0 + (-4) = -4	No buttons			
2	0 – (+4) = -4	No buttons	 (4 zeros)		
3	(+2) + (+4)= -2				
4	(+2) – (+4) = -2		 (2 zeros)		
5	(+2) – (+ 4)= + 6				
6	(+2) – (-4) = +6		 (4 zeros)		
7	(-2) + (+4) = +2				

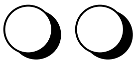
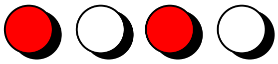
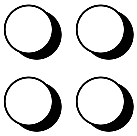
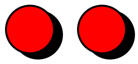
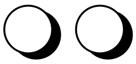
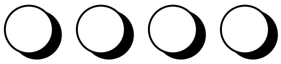
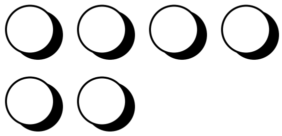
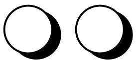
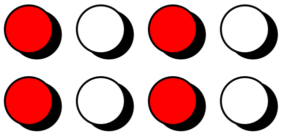
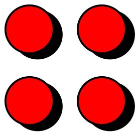
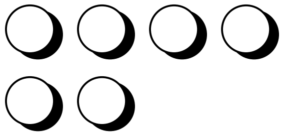
Case	Expression	Start	Buttons added	Buttons taken away	Buttons left
8	(-2) – (-4) = +2		 (2 zeros)		
9	(-2) + (-4) = -6				
10	(-2) – (+4) = -6		 (4 zeros)		

Table 1

Approach 2 : Balloon model

Materials Required: Overhead projector, markers, transparency and the printout of the coloured transparency of the air balloon, helium balloons and sandbags.

Details of the Activity:

1. Outline the model at the beginning of the lesson including explanation of the meaning of "+1" and "-1" and the meaning of addition and subtraction in this model. The teacher may state the importance of choosing suitable signs in usual practice, that is, "+1" represents the balloon moving one unit upwards. The meaning of addition and subtraction should be explained similarly as in the Magnetic-button model.

2. Divide students into groups. Distribute Worksheet 5.1. Help students to formulate mathematical expressions in one of the cases in the Worksheet.

3. Ask students to discuss the various cases in Table 5.1 and write their guesses of the mathematical expressions and the result of each case in the Table.

4. Invite some group representatives to present their findings in Table 5.1 and lead students to check their answers (refer to **Notes for Teachers** for the answers).

5. Guide students to observe that taking off helium balloons is equivalent to hooking on the same amount of sandbags, i.e. in mathematical expression: $a-(+b)=a+(-b)$ for any positive integers a and b. Besides, taking off sandbags is equivalent to hooking on the same amount of helium balloons. Similar conclusion can be arrived: $a-(-b)=a+b$ for any positive integers a and b.

6. Draw students' attention that rising 2 units is the same as moving 2 units towards the right hand side on the number line. Similarly, falling 6 units is the same as moving 6 units towards the left hand side on the number line as shown in Figure 5.1.

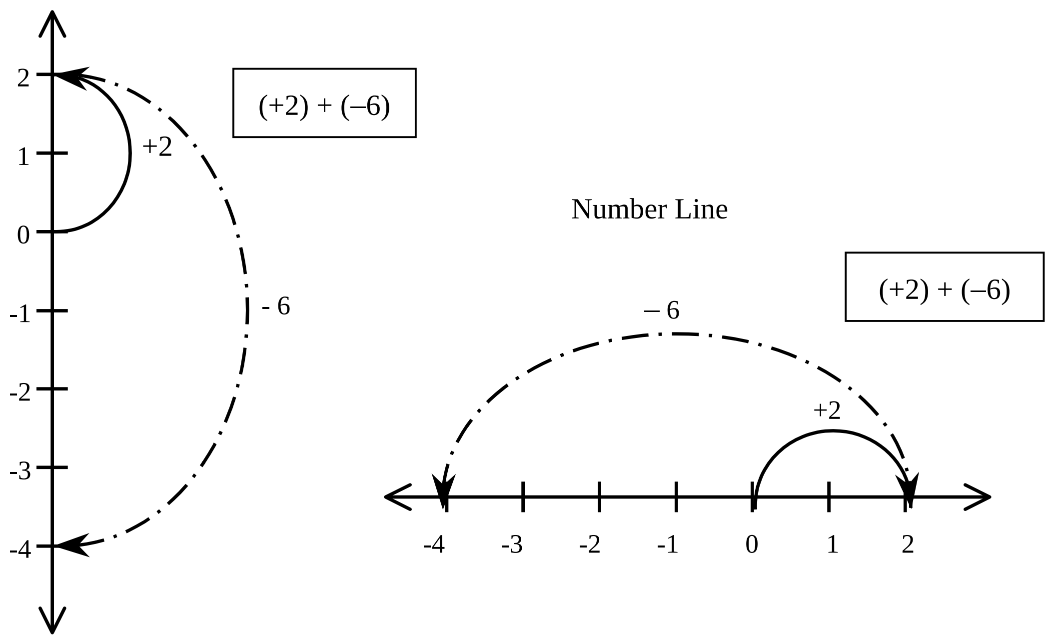
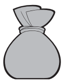


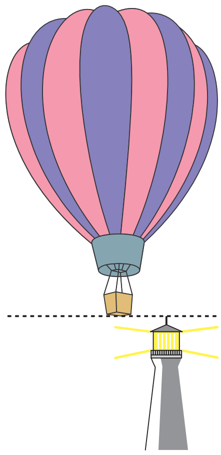
Figure 5.1

7. Guide students to observe the patterns in operating directed numbers as in the case of magnetic-button model. An example to demonstrate the commutative law is suggested as follow. Ask students to work out (+2) + (– 4) and (–4) + (+2). Although in the model the two situations are different, the results are the same. The hot air balloon rising 2 units followed by falling 4 units is different from the case that the balloon falls 4 units followed by rising 2 units. Students are expected to generalize the commutative law $a + b = b + a$. The teacher can go further in the case of subtraction by discussing (+2) – (+4) and (+4) – (+2) to show that the subtraction over directed numbers is NOT commutative.

8. Throughout the discussion of the balloon model, the teacher should help students to perform addition and subtraction of directed numbers on a number line. That is, adding a positive number x means moving x units to the right-hand side whereas subtracting a positive number x means moving x units to the left-hand side. On the other hand, adding a negative number $-x$ means moving x units to the left-hand side whereas subtracting a negative number $-x$ means moving x units to the right-hand side.

Worksheet 5.1 ∶ Case Study - Balloon Model

Suppose that it is a calm day and hot air balloon is tethered at the top of a tower, which is considered to be the zero level. If you hook a sandbag  on your balloon, it will go down, say **one unit below** the top of the tower. On the other hand, if you hook a helium balloons  on the balloon, it will rise, say **one unit above** the top of the tower.



zero level

Now assume that the hot air balloon is right at the top of the tower, having equal number of helium balloons and sandbags hooked on it.

Case	Content	Mathematical Expressions
1	Suppose that you hook on 3 helium balloons and then 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
2	Suppose that you hook on 3 helium balloons and then 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
3	Suppose that you hook on 3 sandbags and then 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
4	Suppose that you hook on 3 sandbags and then 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) + ( ) = ( )
5	Suppose that you hook on 3 helium balloons and then take off 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
6	Suppose that you hook on 3 helium balloons and then take off 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
7	Suppose that you hook on 3 sandbags and take off 5 helium balloons, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )
8	Suppose that you hook on 3 sandbags and then take off 5 sandbags, what will be the final position of the balloon? Above/Below the top of the tower? Where?	( ) – ( ) = ( )

Table 5.1

Worksheet 5.1

(II) Multiplication

Activity 1 : Observation of Pattern in Multiplying Directed Numbers

1. *To illustrate that a negative number times a positive number is negative*.

   Firstly, ask students to complete the following:

   $4 \times 4 =$ ________

   $3 \times 4 =$ ________

   $2 \times 4 =$ ________

   $1 \times 4 =$ ________

   $0 \times 4 =$ ________

   Ask students to observe the pattern for the trend in each case (decreasing by 4 each time) and then raises the question “If the pattern continues, what answer will you expect？”

   Write down the following and ask students to complete:

   $-1 \times 4 =$ ________

   $-2 \times 4 =$ ________

2. *To illustrate that a positive number times a negative number is negative*.

   Similarly, ask students to complete the following：

   $4 \times 4 =$ ________

   $4 \times 3 =$ ________

   $4 \times 2 =$ ________

   $4 \times 1 =$ ________

   $4 \times 0 =$ ________

   Ask students to observe the pattern for the trend in each case (decreasing by 4 each time) and then raises the question “If the pattern continues, what answer will you expect？”

   Write down the following and ask students to complete:

   $4 \times (-1) =$ ________

   $4 \times (-2) =$ ________

3. *To illustrate that a negative number times a negative number is positive*.
   In this case, use a negative number as the multiplier.
   After the previous discussion, students should be able to fill in the following blanks:
   4 × (-4) = _______
   3 × (-4) = _______
   2 × (-4) = _______
   1 × (-4) = _______
   0 × (-4) = _______
   Write down the following and ask students to complete :
   -1 × (-4) = _______
   -2 × (-4) = _______

The following questions can be discussed in the class：

a) (i) What will be the sign of the product of a negative number and a positive number? Is there a difference if the negative number is the multiplier or multiplicand? Why?
   (ii) Can you give a daily-life example to illustrate the meaning of that?

b) (i) What will be the sign of the product of a negative number and a negative number? Why?
   (ii) Can you give a daily-life example to illustrate the meaning of that?

4. The teacher should emphasize that this activity is only an illustration but <u>NOT</u> a formal proof. After this activity, students should have an intuitive idea of the results in multiplying directed numbers. However, they may not be able to explain these results especially for the case of "a negative number times a negative number is positive".

5. As it is anticipated that most students will be unable to give daily life examples for the questions a) and b) in point 3, the teacher can then introduce Activity 2 to guide students to explain the results in daily-life contexts.

Activity 2 : Case Studies

1. Divide students into groups and distribute Worksheet 5.2. Briefly explain the meanings of directed numbers and the operations in various models. For example, representing one helium balloon as "+2", one sandbag as "-3" and hooking as "+" and removing as "–" in the Balloon model whereas in the Mileage model, a car travelling east will have a positive velocity while a car travelling west will have a negative velocity. The teacher can use concrete objects including working on the vertical number line to help students build up an intuition of the resulted position.

2. Then select 2 models out of the 4 given in Worksheet 5.2 for group discussion. Students are expected to answer the questions in the Worksheet.

3. Ask some group representatives to present their findings and the teacher then summarizes the manipulation results in these models.

4. Use a number line to simulate the mileage model or the balloon model (refer to **Notes for Teachers** for the details).

5. By referring to the manipulation in the models, guide students to conclude that:

   i) A positive number denotes an increase in quantity (e.g. a rise in height/time in the future) while a negative number denotes a decrease in quantity (e.g. a fall in height/time in the past)

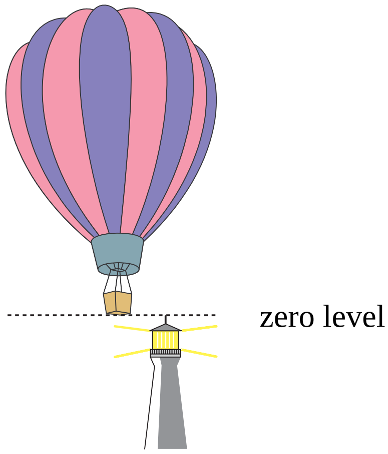
   ii) For any two positive numbers *a* and *b*, $(+a) \times (+b) = a \times b$
   $(+a) \times (-b) = -a \times b$
   $(+a) \times (+b) = -a \times b$
   $(-a) \times (-b) = a \times b$

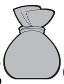
   i.e. the product of two directed numbers of the same sign is positive and of different signs is negative.

Activity 3: Formal proof of the product of two directed numbers of the same sign is positive and of different signs is negative. (For able students only)

Worksheet 5.2 : Case Study

(1) **Balloon model**



Suppose that it is a calm day and your hot air balloon is tethered at the top of a tower, which is considered to be the zero level. If you hook a sandbag  on your balloon, it will go down, say **three units below** the top of the tower. On the other hand, if you hook a helium balloon  on the balloon, it will rise, say **two units above** the top of the tower.

Use directed numbers to represent the following situations:

1. two units above the top of the tower _____________
2. three units below the top of the tower _____________

Now assume that you start the balloon at the top of the tower having certain number of helium balloons and sandbags hooked on it for each of the following cases:

Case	**Do you go up or down? Please tick.**		**Denote your final position by a directed number.**
	UP	**DOWN**	
1. Hook 3 helium balloons.			
2. Hook 2 sandbags.			
3. Remove 3 helium balloons.			
4. Remove 2 sandbags.			

Table 5.2

Write down numerical expressions to show how you get the results in the last column in Table 5.2.

1. ( ) × ( ) = ______________________________

2. ( ) × ( ) = ______________________________

3. ( ) × ( ) = ______________________________

4. ( ) × ( ) = ______________________________

(2) **Earning and spending money model**

Suppose you find a summer job that you will earn \$150 per day. But if you are on vacation, you will spend \$100 a day.

Case 1: You are doing a summer job and earning money.
3 days later, are you richer or poorer than you are today? By how much?

Case 2: You are on vacation and are spending money.
3 days later, are you richer or poorer than you are today? By how much?

Case 3: You are back at your job and earning money.
3 days ago, are you richer or poorer than you are today? By how much?

Case 4: You are back on vacation and are spending money.
3 days ago, are you richer or poorer than you are today? By how much?

By using directed numbers to represent the amount of money you earn / spend and the number of days elapsed / in the future, write down numerical expressions to show how you get the results for the above four cases.

1. ( ) × ( ) = ________________

2. ( ) × ( ) = ________________

3. ( ) × ( ) = ________________

4. ( ) × ( ) = ________________

(3) **Filling / draining a swimming pool**

Case 1: You are filling the pool at a rate of 4 litres per second.
When comparing with the present situation, how does the amount of water change in 5 seconds? Increase or decrease? By how much?

Case 2: You are draining the pool at a rate of 4 litres per second.
When comparing with the present situation, how does the amount of water change in 5 seconds? Increase or decrease? By how much?

Case 3: You are filling the pool at a rate of 4 litres per second.
When comparing with the present situation, how did the amount of water change 5 seconds before? Increase or decrease? By how much?

Case 4: You are draining the pool at a rate of 4 litres per second.
When comparing with the present situation, how did the amount of water change 5 seconds ago? Increase or decrease? By how much?

By using directed numbers to represent the amount of water you fill the pool / drain from it and the time elapsed / in the future, write down numerical expressions to show how you get the results for the above four cases.

1. ( ) × ( ) = ____________________

2. ( ) × ( ) = ____________________

3. ( ) × ( ) = ____________________

4. ( ) × ( ) = ____________________

(4) **Mileage model**

Suppose that you are standing on a road and you measure mileage to the east as positive and to the west as negative. So you are at zero, and a town one kilometer east is at +1 km, while a town two kilometer west is –2 km.

At the present moment, there is a car passing you.

Case 1: Suppose that the car passes you going east at 75km/h. Where will it be in two hours?

Case 2: Suppose that the car passes you going east at 75km/h. Where was it two hours ago?

Case 3: Suppose that the car passes you going west at 75km/h. Where will it be in two hours?

Case 4: Suppose that the car passes you going west at 75km/h. Where was it two hours ago?

Write down numerical expressions to show how you get the results for the above four cases.

1. (   ) × (   ) = ______________________
2. (   ) × (   ) = ______________________
3. (   ) × (   ) = ______________________
4. (   ) × (   ) = ______________________

(III) Division: Case study — Thermometer model

1. Take a thermometer to class and ask students the range of the temperature that can be read. Then discuss with students the following situations:
   (i) Suppose that, on average, the temperature increases two degrees for each hour and the temperature at the present moment is 0 degree.
      (a) What will be the temperature after 3 hours?
      (b) How long will it take to increase to 14 degrees?
   (ii) Suppose that, on average, the temperature drops three degrees for every hour and the temperature at the present moment is 0 degree.
      (a) What would be the temperature 4 hours ago?
      (b) How long will it take to drop to -15 degrees?
      (c) When did you get the temperature 24 degrees?
   (iii) Suppose that the temperature at the present moment is 0 degree. If the temperature becomes -24 degrees in 6 hours, on average, what is the change in the temperature in one hour?

2. Ask students to use directed numbers to represent temperature drop and rise, the time elapsed and the time in the future.

3. Invite students to write down numerical expressions on the blackboard to illustrate how they get the results for the questions given in point 1. For less able students, the teacher can use a big thermometer model to show the decrease (or increase) of temperature step by step to enable students to visualize the results.

4. Guide students to conclude the following rules in manipulating division:

   For any two positive numbers *a* and *b*, $\frac{(+a)}{(+b)} = \frac{a}{b}$

$$\frac{(+a)}{(-b)} = -\frac{a}{b}$$

$$\frac{(-a)}{(+b)} = -\frac{a}{b}$$

$$\frac{(-a)}{(-b)} = \frac{a}{b}$$

5. Remind students that when they multiply the denominator to the right hand side, they can obtain exactly the rules in multiplying directed numbers. In other words, the rules in manipulating division can be derived directly from those in multiplication.

Part B：A Card Game

Details of the Activity:

1. Divide students into groups of 4 and distribute each group a pile of cards and Worksheet 5.3.

2. Explain the rules of the game to students.

   (a) Shuttle the cards and place the top four cards with face up on the desk.
   (b) With the four numbers shown on the cards, use "+", "–", "×", "÷" or "( )" to form a numerical expression equal to 24.
   For example,

Four directed numbers	Expression that can be formed
-4, -3, -2, -1	(-1) × (-2) × (-3) × (-4)
	((-1) + (-2) + (-3)) × (-4)

   (c) In each group, the student who uses the shortest time to get the correct answer will be the winner. He / She will score 1 point. Record the result in Table 5.3 in Worksheet 5.3.
   (d) Continue the game for the remaining cards in the pile.
   (e) If there is no response from the group for the four directed numbers shown, students still need to shade these set of numbers in the first column of Table 5.3.

3. Invite the representative of each group to write down the combinations of the four directed numbers. Other group members can also be invited to add on some other combinations of the same 4 directed numbers.

4. Invite students to write down those 4 numbers that the numerical expression cannot be found. Discuss with students to countercheck the numbers that the numerical expressions cannot be found.

Worksheet 5.3： A card game

Four directed numbers	Expression	Put "1" point for the winner under appropriate column.			
		Student			
		1	2	3	4

Table 5.3

Part C : A Card Game

Details of the Activity:

1. Follow the discussion of adding the number of combinations for the same 4 directed numbers in Part B of the Activity, challenge students to find **the total number of ways** to form the numerical expression equal to 24.

2. Explain the rule of the game to students:
   i) With the four directed numbers given and the arithmetic symbols "+", "–", "×", "÷" or "( )" as in the Worksheet 5.4, students mark the expressions for the numbers. For the example given in Part B, there are 2 distinct ways in forming such kind of expressions for the directed numbers –1, –2, –3, and –4.
   ii) Expressions of the forms $a\times b\times c\times d$, $b\times a\times c\times d$, $a\times c\times b\times d$, etc are treated as the same and should be counted only once, similarly for addition.
   iii) Expressions of the forms $a\div b\times c\times d$, $c\div b\times a\times d$, $d\div b\times a\times c$, $a\times c\times d\div b$ and $d\div b\times a\times c$ are considered as the same and will be counted once.
   iv) Expressions of the forms $(a-b)\times c\div d$, $(a-b)\div d\times c$, $c\div d\times(a-b)$ are treated as the same.

3. Explain to students that the rules are for convenient sake and can be changed, as there are no strong mathematical reasons for the above setting.

4. Distribute Worksheet 5.4 to students to record the results and students are expected to complete the task within 20 minutes.

5. Discuss with students the answers they get.

Worksheet 5.4：A card game

Four directed numbers	Expression(s)	Total number of expression(s)
-2, -2, -8, -10		
-2, -3, -7, -9		
-3, -4, -5, -10		
-3, -5, -7, -9		
-4, -4, -4, -10		
-4, -7, -8, -10		
-5, -5, -5, -5		
-5, -6, -8, -10		
-6, -7, -8, -10		
-8, -8, - 8, -10		

Table 5.4

Notes for Teachers :

Part A

(I) Addition and Subtraction

Approach 1 : "Magnetic-button" model

1. After the activity, students can arrive at the following conclusion.
   For any two positive numbers a and b,

Addition	*Subtraction*
$(+a)+(+b)=a+b$	$(+a)-(+b)=a-b$
$(+a)+(-b)=a-b$	$(+a)-(-b)=a+b$
$(-a)+(+b)=-a+b$	$(-a)-(+b)=-a-b$
$(-a)+(-b)=-a-b$	$(-a)-(-b)=-a+b$

(II) Multiplication

Activity 1 : Observation of patterns

1. The teacher can carry out the activity in the form of a contest to encourage students to answer questions.

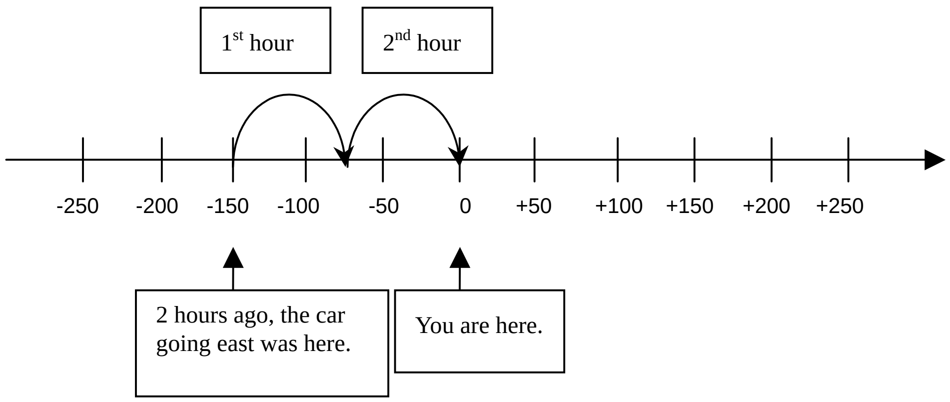
Activity 2 : Case Study

1. Answers to the case study:

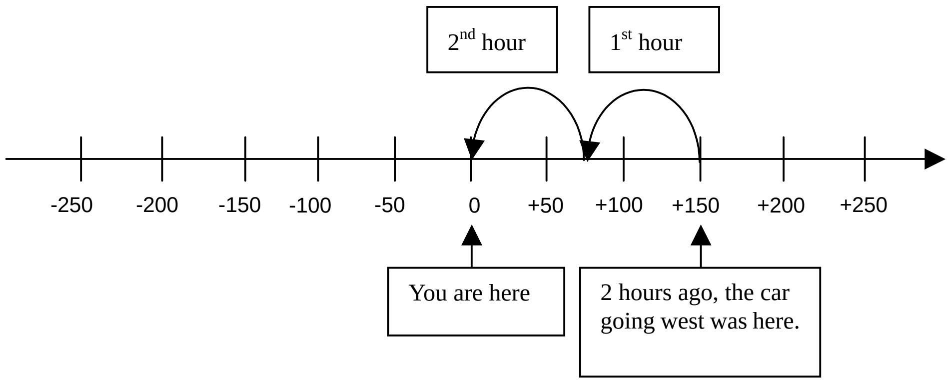
Model	Mathematical expressions
Balloon model	1. (+2) × (+3) = +6 2. (-3) × (+2) = -6 3. (+2) × (-3) = -6 4. (-3) × (-2) = +6
Earning and spending money	1. (+150) × (+3) = +450 2. (-100) × (+3) = -300 3. (+150) × (-3) = -450 4. (-100) × (-3) = +300
Filling / Draining a swimming pool	1. (+4) × (+5) = +20 2. (-4) × (+5) = -20 3. (+4) × (-5) = -20 4. (-4) × (-5) = +20
Mileage model	1. (+75) × (+2) = +150 2. (+75) × (-2) = -150 3. (-75) × (+2) = -150 4. (-75) × (-2) = +150

2. The illustration with a number line to explain the idea of multiplication of directed numbers in the mileage model is as follow:

Case 2:



Case 4:



(III) Division

1. Answers to the case study：

Thermometer model	Mathematical expressions
(I)	2 × 3 = 6 (degrees)
	$\frac{(+14)}{(+2)} = 7$ (hours)
(II)	(-3) × (-4) = 12 (degrees)
	$\frac{(-15)}{(-3)} = 5$ (hours)
	$\frac{(+24)}{(-3)} = -8$; 8 hours before
(III)	$\frac{(-24)}{(+6)} = -4$; drops 4 degrees per hour

Activity 3 : Formal Proof (Only for able students)

Prerequisite Knowledge : Distributive law over addition

Prove that the product of two directed numbers of the same sign is positive and of different signs is negative.

Proof:

For any positive numbers a and b,

$$a + (-a) = 0$$
$$[a+(-a)]\times b = 0\times b$$
$$a\times b + (-a)\times b = 0$$
$$\therefore\ (-a)\times b = -a\times b \quad \text{and} \quad (-a)\times b < 0$$

Similarly, we can show that the product $a\times(-b)$ is negative and equals $-a\times b$.

On the other hand,

$$a + (-a) = 0$$
$$[a+(-a)]\times(-b) = 0\times(-b)$$
$$a\times(-b) + (-a)\times(-b) = 0$$
$$\therefore (-a)\times(-b) = -\ a\times(-b) \text{ and } (-a)\times(-b) > 0$$

Also,

$$a\times(-b) + (-a)\times(-b) = 0$$
$$-a\times b + (-a)\times(-b) = 0$$
$$\therefore\ (-a)\times(-b) = a\times b \text{ and } (-a)\times(-b) > 0$$

Part B and C

1. The value of the numerical expression is set at 24 since it is the smallest positive number that has the most factors lying between -10 and -1. The teacher may change it to other values.

2. The details of the game 24 for positive integers from 1 to 10 can be found in the following book:
王紹川主編 胡大同、王朝輝副主編 (1994)。《數學 24 遊戲》。 中國：煤炭工業出版社
The teacher needs to make some modification to the results given in the book to fit the case of directed numbers.
For examples,

In the book		Modification	
Four positive integers	**Expression(s)**	**Four integral directed numbers**	**Expression(s)**
2, 2, 8, 10	(10 – 2) × 2 + 8	-2, -2, -8, -10	(-10 – (-2)) × (-2) – (-8)
	10 × 2 + 8 ÷ 2		(-10) × (-2) + (-8) ÷ (-2)
2, 3, 7, 9	(7 – 2) × 3 + 9	-2, -3, -7, -9	(-7 – (-2)) × (-3) – (-9)
	3 × (9 + 7) ÷ 2		Nil
3, 4, 5, 10	10 ÷ 5 × 3 × 4	-3, -4, -5, -10	(-10) ÷ (-5) × (-3) × (-4)
	(5 – 3) × 10 + 4		(-5 – (-3)) × (-10) – (-4)
	3 ÷ 5 × 4 × 10		Nil
	4 ÷ 5 × 10 × 3		Nil

3. Suggested answers to Worksheet 5.4:

Four directed numbers	Expression(s)	Total number of expression(s)
-2, -2, -8, -10	$\left(-10-(-2)\right)\times(-2)-(-8)$ $(-10)\times(-2)+(-8)\div(-2)$	2
-2, -3, -7, -9	$\left(-7-(-2)\right)\times(-3)-(-9)$	1
-3, -4, -5, -10	$(-10)\div(-5)\times(-3)\times(-4)$ $\left(-5-(-3)\right)\times(-10)-(-4)$	2
-3, -5, -7, -9	$-\left[(-3)+(-5)+(-7)+(-9)\right]$ $(-9)\times(-5)-(-3)\times(-7)$ $\left((-9)+(-3)\right)\times\left((-7)-(-5)\right)$	3
-4, -4, -4, -10	$(-4)\times(-10)-(-4)\times(-4)$	1
-4, -7, -8, -10	Nil	0
-5, -5, -5, -5	$(-5)\times(-5)-(-5)\times(-5)$	1
-5, -6, -8, -10	$(-5)\times(-8)+(-10)+(-6)$ $\left((-8)-(-5)\right)\times(-10)+(-6)$ $\left((-8)\times(-5)\right)\div(-10)\times(-6)$	3
-6, -7, -8, -10	$\left(-7+(-6)-(-10)\right)\times(-8)$ $\left(-8-(-6)\right)\times(-7)-(-10)$	2
-8, -8, -8, -10	$\left(-10-(-8)\right)\times(-8)-(-8)$	1

Appendix: Reference:

Web sites:

1. History of negative numbers
   http://www.edp.ust.hk

2. For addition and subtraction,
   http://riceinfo.rice.edu/armadillo/Algebra/Lessons/Negative
   http://www.col-ed.org/cur/math

3. For multiplication and division,
   http://riceinfo.rice.edu/armadillo/Algebra
   http://forum.swarthmore.edu/dr.math
   http://www2.ncsu.edu/unity/lockers/users/f/felder/public

Books:

1. Stein, Sherman K. (1996). *Strength in numbers: discovering the joy and power of mathematics in everyday life*. New York: John Wiley.

2. 王紹川主編 胡大同、王朝輝副主編 (1994)。《數學 24 遊戲》第 1 版。
   中國：煤炭工業出版社。

Appendix: Reference

Negative Numbers

The concept of negative numbers, addition, subtraction of positive numbers and negative numbers were introduced in Chapter 8 “Equation” of the book *Nine Chapters of Mathematical Art* in China. In some questions, the number of sold items was regarded as positive (income) while the number of bought-in items was regarded as negative (spending); remaining amount as positive and overdue amount as negative. In calculating food problems, positive means adding in while negative means taking away. These two words are still using in similar context nowadays.

In the chapter of “Equation”, the addition of positive and negative numbers is called “Zheng Fu Shu” (Arithmetic of positive/negative numbers). However, multiplication and division of them came much later. In 1299, Zhu Shi-jie in his book *The Enlightenment of Mathematics* wrote eight rules about addition and subtraction of positive and negative numbers, and these rules were more specific than that in *Nine Chapters of Mathematical Art*. There was a passage in *Ming Cheng Chu Duan* saying that “Two numbers of the same sign multiplied together give positive result, whereas two numbers of opposite signs multiplied together give negative result.” this is the same as:

$(+a)\times(+b) = +ab$, $(-a)\times(-b) = +ab$,

$(+a)\times(-b) = -ab$, $(-a)\times(+b) = -ab$,

These kinds of rules were the earliest Chinese records regarding multiplication of positive and negative numbers. At the end of Song dynasty, Li Ye invented the use of a slash at top of a number to represent a negative number. The introduction of negative numbers is one of the most excellent inventions in the Ancient China.

The earliest Indian, who introduced negative number, was Brahmagupta (approx. 598 – 665) at about 628 A.D.. He proposed the operation rules of negative numbers and abbreviated it by putting a small dot or small circle on it. The first European, who knew and proposed negative numbers, was the Italian Fibonacci (1170 – 1250). When he tried to solve a profit problem, he said “I shall prove that this question has no answer unless one can carry debt.” Although in the 15^{th} century Chuquet (1445 – 1510) and in the 16^{th} century Stifel (1553) both discovered negative numbers, but they still considered that those numbers as ridiculous numbers. Cardan (1545) found the negative root of equation but once again he put it as “unreal” number. Viete knew the existence of negative numbers but totally discarded it. Descartes accepted negative numbers partially. He named the negative roots of equation as “unreal” root as they were smaller than zero.

Harriet (1560 – 1621) accidentally put negative numbers alongside of an equation and use "–" sign to represent them, but he did not accept negative numbers. Bombelli (1526 – 1572) gave a more precise definition of negative numbers. Stevin used positive and negative coefficients in equations and accepted negative roots. Girard (1595 – 1629) put negative numbers in the same status as of positive numbers and used "-" sign to represent negative sign. Generally speaking, European accepted negative numbers in the16 – 17th centuries but the progress was rather slow.

Note: This article is translated from the web-page of http://www.edp.ust.hk