Exemplar 2: Exploration of Fibonacci Sequence

Dimension : Number and Algebra

Learning Unit : Formulating Problems with Algebraic Language

Key Stage : 3

Materials Required : Part B – Pictures of some flowers in nature
Part C – A worksheet designed with spreadsheet software such as *Microsoft Excel*

Prerequisite Knowledge :
(1) Basic ability in observing the patterns of simple number sequences
(2) Use algebraic symbols to represent the general terms in these sequences.

Key Features :
This exemplar consists of three parts that cater for different learning abilities of students as shown in the following table. All the three parts are in the Foundation Part of the same learning unit in the Syllabus.

Part	Less able students	Average students	More able students
A	✓	✓	✓
B		✓	✓
C			✓

Remark : ✓ represents the part(s) that can be participated by students when they start to learn the captioned topic.

Part A：
This part is to let students discover Fibonacci Sequence (1, 1, 2, 3, 5, 8,.….) from daily life experience. There are two activities — studying wall patterns and climbing a flight of stairs if they can move either one or two steps forward at any point. The first one is designed for less able students while the second for average and more able students. Through the activity, students can develop a sense of recursion in problem solving.

Part B：

This part is a project work that requests students to investigate how Fibonacci Sequence is related to the Nature. Students will appreciate the aesthetic nature of mathematics.

Part C：

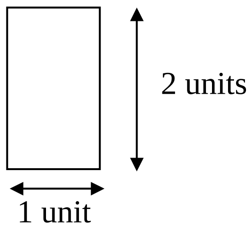
Students are asked to study the ratio of two successive terms in Fibonacci Sequence and find out the limiting value of the ratios by using the software *Microsoft Excel*.

Description of the Activity :

Part A :

Activity 1 (For less able students)

Materials required : Identical blocks each has a length of 1 unit and a height of 2 units

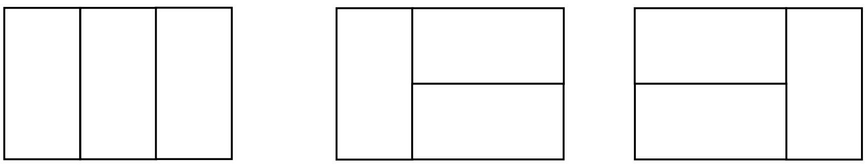


1. Distribute students a certain number of blocks.

2. Ask students to use the blocks to build a wall two units tall and of various lengths such as 1 unit, 2 units, 3 units and so on, and count the number of wall patterns obtained.

3. Allow students to work in pair.

4. The teacher can demonstrate how to construct a wall of length 3 units and height 2 units as an example. Three different wall patterns can be obtained:



5. Distribute Worksheet 2.1 to students for activity 1 and ask them to complete the worksheet.

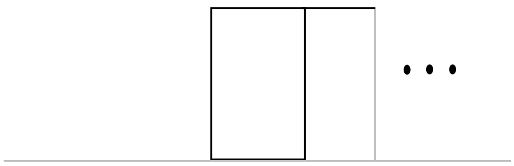
6. Invite students to report what they have found from the results obtained in the worksheet.

7. Discuss the following questions with students:

   (a) What do you observe from the number of wall patterns?
   (b) How many wall patterns are there for a wall of length 10 units and height 2 units? Explain how you get the answer.
   (c) Suppose T(n) denotes the number of patterns for walls of height 2 units and length n units.
   (i) What are the meanings of T(n – 2) and T(n – 1)?
   (ii) Can you use the algebraic notation T(n – 2), T(n – 1) and T(n) to express the relation in the wall patterns?

8. Further ask students to explain why condition "T(n) = T(n – 1) + T(n – 2) for n > 2 and n is a natural number" holds in general in the activity. Some guideline is suggested below for reference:
   (a) The teacher may guide students to consider what happens at the left end of the wall. There are two different cases.

Case 1:



There is a block standing upright on its end.

Case 2:

There are two blocks with one lying on the other to make a height of 2 units on its end.

   (b) Questions to be discussed:
      (i) How many wall patterns are there if we start a wall of length n units with a single upright block on its end?
      Answer: There are T(n – 1) wall patterns since we need patterns for walls of length n – 1 units to complete it.

(ii) How many wall patterns are there if we start a wall of length n units with the two flat blocks on top of each other on its end?
Answer: There are T(n – 2) wall patterns since we need patterns for walls of length n – 2 units to complete it.

(c) The teacher then helps students conclude that the number of patterns of a wall of length n is the sum of patterns of a wall of length (n – 1) units and the number of patterns of a wall of length (n – 2) units i.e. $T(n) = T(n-1) + T(n-2)$.

9. The teacher finally introduces students the numbers 1, 2, 3, 5, 8, ……which are in Fibonacci sequence.

Worksheet 2.1 : Activity 1

To build a wall two units high and of various length.

Complete the following table

Length of the wall (units)	Wall patterns	No. of wall patterns
1		
2		
3		
4		
5		
6		
7		

Observation: Describe the patterns about the number of wall patterns constructed, if any.

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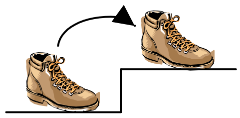
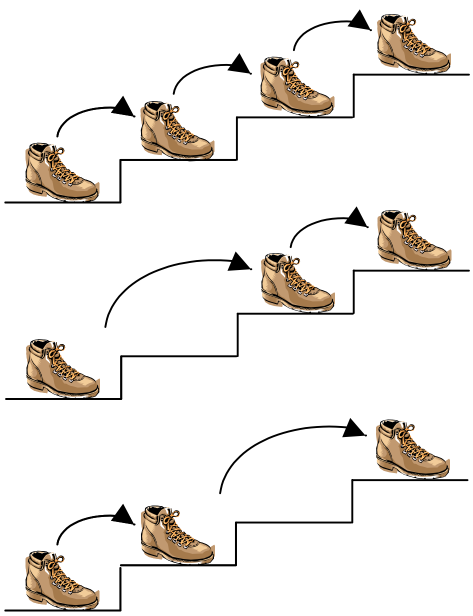
Activity 2 (For average and able students)

1. The teacher describes the following situation to students at the beginning of the lesson：
   As I want to get a little more exercises these days, I try and take the stairs rather than the elevator whenever I go. In this morning, I was in a hurry, so I leaped two stairs at once or stepped onto the next stair at a time. Now suppose that I mix these two kinds of action. In how many ways can I get up a flight of n stairs?

2. The teacher first discusses with students the number of ways that he/she can go up 1 step, 2 steps and 3 steps of a flight of stairs as shown in the following table.

No. of steps	Different ways to go up the stairs	Total no. of ways
1		1
2		2
3		3

Ask students to work in pairs to count the number of possible ways to go up a flight of 4 stairs, 5 stairs, 6 stairs and so on. Record the results in the table in Worksheet 2.2 for activity 2. Seek for a pattern of the number of ways. Invite students to describe their findings to the class. Encourage them to explain their observations and the teacher can make comment at appropriate time.

3. Give students adequate time to generalize the result to a flight of n stairs and use algebraic symbols to represent it. Answer question 2 in Worksheet 2.2 for activity 2.

4. After checking the answers in the worksheet, the teacher may invite some students to explain why the condition "$T(n) = T(n-1) + T(n-2)$ for $n > 2$ and n is a natural number" holds in general in the activity. That is, the number of ways of climbing a flight of n stairs is the sum of the number of ways of climbing a flight of $(n-1)$ stairs and the number of ways climbing a flight of $(n-2)$ stairs.

5. The following questions are suggested for discussion:

   (a) Before stepping on the nth stair, which step do you reach if
      (i) you leap two steps at once for your last step;
      (ii) you do not leap any step for your last step?
   (b) How many ways for you to go up a flight of n stairs if you move two steps forward in the last step?
   (c) How many ways for you to go up a flight of n stairs if you move one-step forward in the last step?
   (d) How many ways for you to go up a flight of n stairs if you move one or two steps forward at any point?

6. The teacher then introduces Fibonacci Sequence and concludes that the problem of counting the number of ways to go up a flight of stairs can be solved recursively by computing the Fibonacci Sequence which is a recursive sequence with the first two values 1 and each successive term obtained by adding together the two previous terms. The sequence is 1, 1, 2, 3, 5, 8, …which can be described by the conditions $T(1) = T(2) = 1$ and $T(n) = T(n-1) + T(n-2)$ for $n > 2$ and n is a natural number.

Exemplar 2

Worksheet 2.2 : Activity 2

Suppose that you climb up a flight of *n* stairs. You can move either one or two steps forward at any point.

1. Complete the following table:

Number of stairs	**Total number of ways**
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

2. Let *n* denote the number of stairs and T(*n*) be the total number of ways climbing a flight of *n* stairs.

   (a) Find the value of T(15).

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   (b) Describe the relation for the pattern you find in the table.

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   (c) Use algebraic symbols such as *n* and T(*n*) to represent the relation obtained in part 2(b).

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Part B :

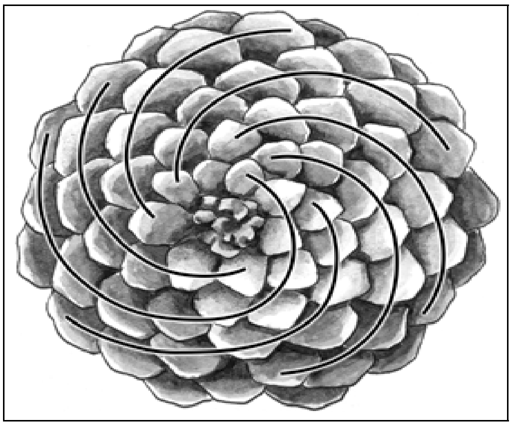
Ask students to do a project for investigating the pattern of Fibonacci Sequence in the Nature such as:

1. The number of growing points in a branching plant

2. The number of petals on flowers

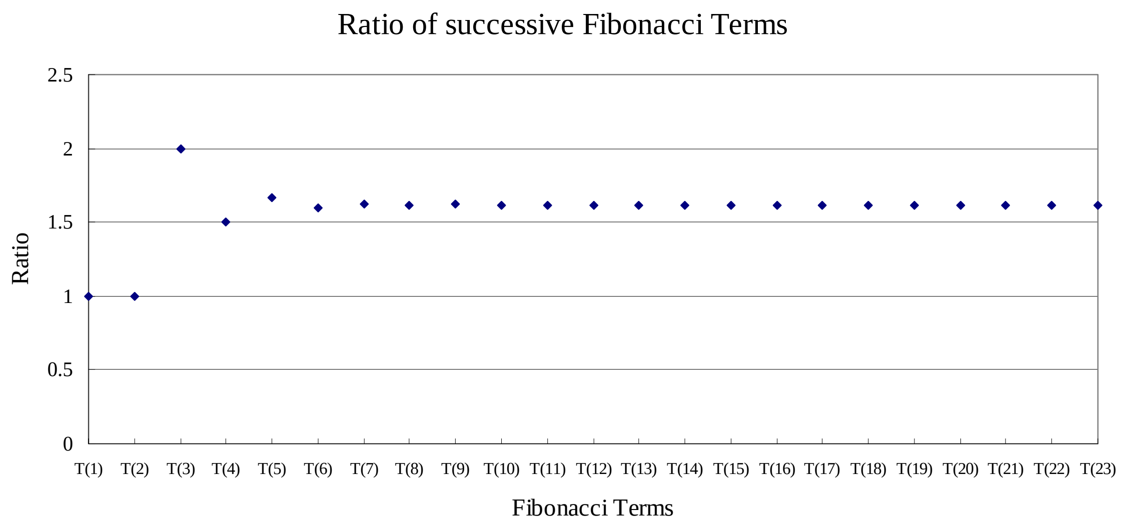
3. The arrangement of seeds on flower heads - the number of spirals curving to the left and to the right

4. The number of spirals of pine cones



Part C :

Ask students to calculate the ratios of two successive terms in Fibonacci Sequence (divide each by the number before it). Plot the ratio of successive Fibonacci terms against the Fibonacci number as shown in the figure below. Students may use *Excel* when plotting graph. Describe what they observe from the graph.



Predict what happens if they take the ratios the other way round, i.e. divide each number by the one following. Is there any relation between the two results obtained? Justify their conjecture graphically.

Notes for Teachers :

Part A :

Activity 1

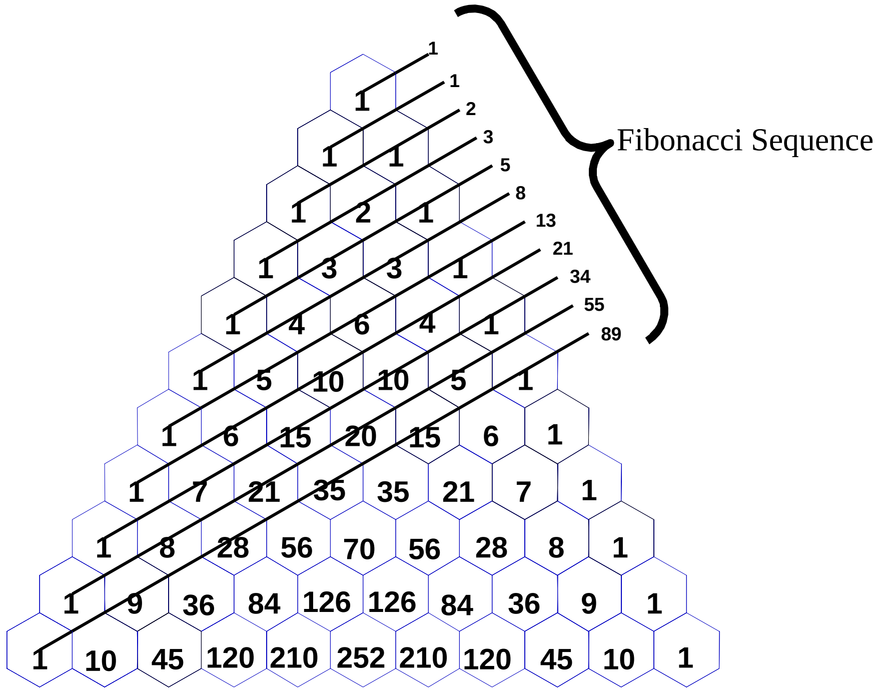
1. The teacher needs to revise with students how to use T(n) to denote the nth term in a sequence before the discussion session given in point 7 in the description of the activity.

2. The numbers of wall patterns are in Fibonacci Sequence which is given by the following conditions:
   (i) T(1) = 1 AND
   (ii) T(2) = 2 AND
   (iii) T(n) = T(n – 2) + T(n – 1) for n > 2 and n is a natural number.

Activity 2

1. The teacher should make sure that students can use a notation T(n) to denote the nth term in a sequence when introducing the Fibonacci Sequence.

2. Suggested follow-up activity for this part: Ask students to search for the Fibonacci Sequence in a Pascal's Triangle, if they have learnt Pascal's Triangle previously.



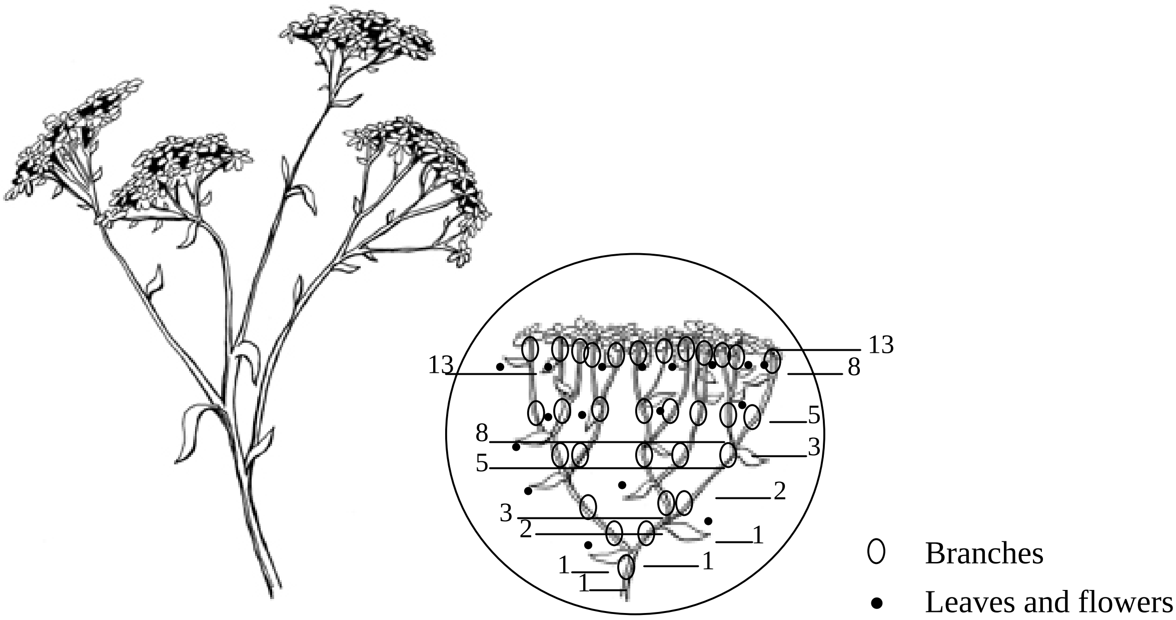
Pascal's Triangle

Part B :

1. This part is a cross-curricular project. Students can integrate the learning experience in Science and Mathematics. A field trip may be arranged for this project.

2. Throughout the activity, students can discover that
   (a) Fibonacci numbers appear in the number of growing points in some branching plants. The teacher can ask students to find out what kind of plant has such feature.

The following figure shows Fibonacci Numbers in the Sneezewort plant.



(b) The number of petals is a Fibonacci Number.

Number of petals	**Species**
2	Begonia
3	Tillandsia, Iris
5	Bauhinia blakeana, Allamanda neriifolia, P. Casandra, Bombax malabaricum, Delonix regia
8	Some delphinium
13	Some marigold, Cineraria
21	Some aster, Some marigold
34	Plantain Lily

Reference information for the flowers can be found in the web site:

http://www.flowerweb.com

Students may collect some specimen of flowers to illustrate the results.

Begonia (2)

Tillandsia (3)

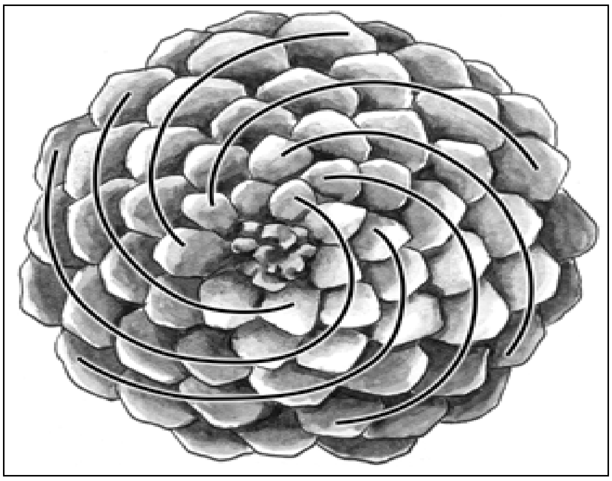
Bauhinia blakeana (5)

Allamanda neriifolia (5)

P. Casandra (5)

(c) The numbers of spirals on seed heads or pine cones curving to the left and to the right are neighbors in Fibonacci Series.

Seed head



Pine cone

3. If the above phenomena of nature has been used in motivating students' interest in the introduction of Fibonacci Sequence in Part A, the teacher may ask students to do the investigation on the relation of the Fibonacci Sequence and the vegetables and fruits.

   Reference information can be found in the web site:
   http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html

Part C :

1. Students are expected to find out that the ratio seems to be settling down to a particular value of approximately 1.61804 in the first case and 0.618034 in the latter case. These two numbers are reciprocal to each other. The number 1.61804 is called the **golden ratio.**

2. The teacher can further ask students to find out how the golden ratio is applied in Art, Architecture and Music. Reference for this can be found in the following website:
   http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibInArt.html#arch
   http://www.indiana.edu/~kglowack/athens/acropolis.html

Reference Materials :

More information about Fibonacci Sequence and Golden Ratio can be found here:

Websites:

1. http://mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html
2. http://mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpi.html#piandfib. This web page explores the relationship between the Fibonacci Sequence and the number π.
3. http://www.sdstate.edu/~wcsc/http/fibhome.html
4. http://forum.swarthmore.edu/dr.math/tocs/golden.high.html
5. http://matheworld.wolfram.com/FibonacciNumber.html. This web page involves some advanced mathematics.

Books :

1. Wells, David G. (1998). *The Penguin Dictionary of Curious and Interesting Numbers*. Penguin Press.

2. Runion, Garth E. (1990). *The Golden Section*. Palo Alto, CA: Dale Seymour Publications.

3. Garland et al. (1987). *Fascinating Fibonacci: Mystery and Magic in Numbers*. Palo Alto, CA: Dale Seymour Publications.

4. Dunlap, Richard A. (1998). *The Golden Ratio and Fibonacci Numbers*. Singapore: World Scientific Publishing Company.

5. Garland, Trudi Hammel, et al. (1998). *Fibonacci Fun: Fascinating Activities with Intriguing Numbers*. Palo Alto, CA: Dale Seymour Publications.

6. Pappas, Theoni. (1989). *The Joy of Mathematics: Discovering Mathematics All Around You*. San Carlos, CA: World Wide Publishers/Tetra.

7. Rockett, Andrew Mansfield. (1992). *Continued Fractions*. Singapore: World Scientific Publishing Company.

8. 吳振奎編著(1993) 。《斐波那契數列》。臺北：九章出版社。