Exemplar 1: Investigation of Lines in a Triangle

Dimension: Measures, Shape and Space

Learning Unit: Simple Introduction to Deductive Geometry

Key Stage: 3

Materials Required: Dynamic Geometry software such as *Geometer's Sketchpad* (later referred as *Sketchpad*)

Key Features:

This exemplar consists of three parts that cater for different learning abilities of students as shown in the following table. Part A belongs to the Foundation Part of the *Syllabus*, part B belongs to the Non-Foundation Part while part C is an enrichment topic that targets for more able students.

Part	Prerequisite Knowledge	Less able students	Average students	More able students
A	• Different kinds of triangles	✓	✓	✓
B	• Basic IT geometric construction skills • Meaning of median		✓	✓
C	• Intuitive idea of deductive reasoning • Conditions for congruent and similar triangles • Properties of similar triangles & parallelograms • Mid-point Theorem and Intercept Theorem.		✓	✓

Remark : ✓ represents the part(s) that can be participated by students when they start to learn the captioned topic.

Part A：

This part is to let students identify and explore properties of some special lines in a triangle, namely medians, altitudes, angle bisectors and perpendicular bisectors in working with four different *Sketchpad* files. Worksheets 1.1A and 1.1B are designed to cater for average and less able students respectively. Students are asked to investigate the properties of these lines in a triangle by measuring

the line segments and angles of the figures in the *Sketchpad* files. The situations in which some of these special lines coincide will also be investigated.

Part B：

Only medians are considered in this part of the activity. Students are required to explore and recognize the concurrent properties of medians and the ratio at which the centroid divides the medians. Worksheet 1.2 is designed for students to follow through the activity with guided questions in discovering the properties. The teacher can ask students to investigate whether the same properties hold for other lines such as altitudes, angle bisectors and perpendicular bisectors in the triangle.

Part C：

Students are asked to give geometric proofs to verify the findings in part B. They are expected to use the results related with triangles to perform simple proofs. The teacher can select different strategies applied in the geometric proofs according to the abilities of students. For example, the teacher may request more able students to use the Mid-point Theorem to do the proof while the others can use the properties of similar triangles with given hints.

Description of the Activity：

Part A：To identify medians, altitudes, angle bisectors and perpendicular bisectors in a triangle.

1. Divide students into groups. Each group is given four Sketchpad files, namely tri1.gsp, tri2.gsp, tri3.gsp and tri4.gsp as shown in Figures 1, 2, 3 and 4 respectively. They are asked to investigate how to construct lines BE or GE; or AD or GD in a triangle.

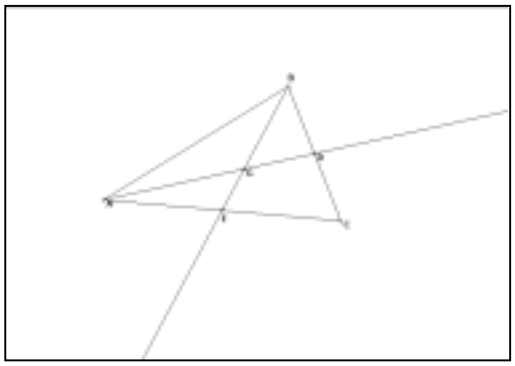


Figure 1

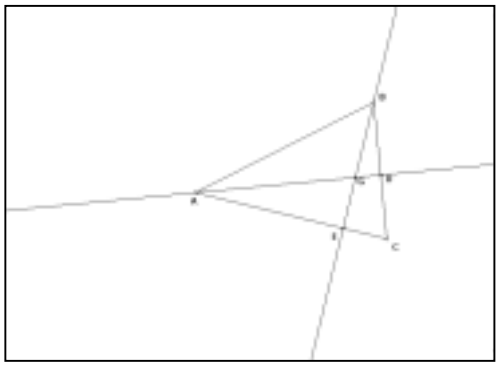


Figure 2

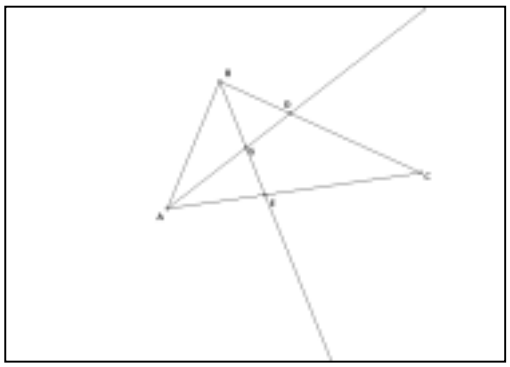


Figure 3

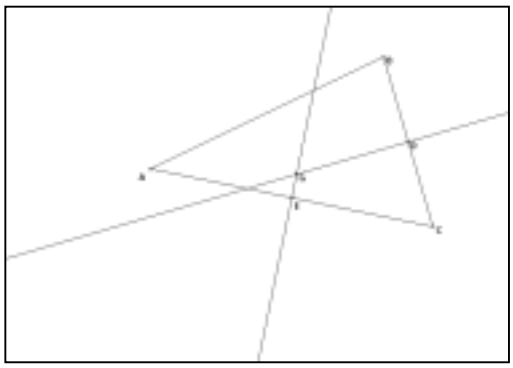


Figure 4

2. Students are asked to change the shape of the triangle by dragging any one of the vertices and observe the changes on the lines. They can measure the sides and the angles formed as shown in Figures 5, 6, 7 and 8 respectively. Write down their findings in Section A of the worksheet. Worksheet 1.1A is designed for average students while Worksheet 1.1B is designed for less able students.

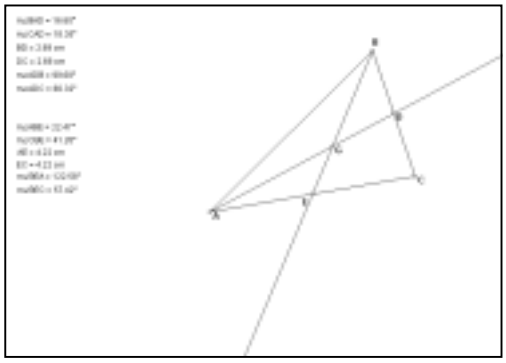


Figure 5

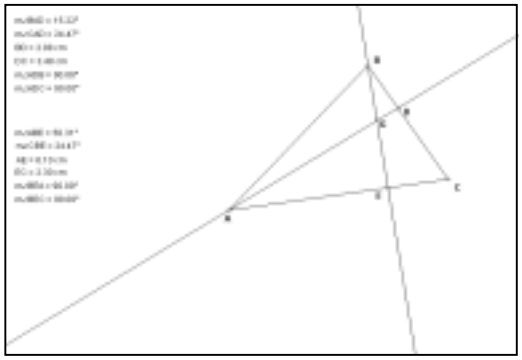


Figure 6

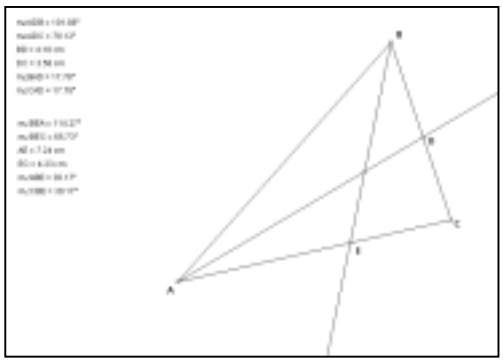


Figure 7

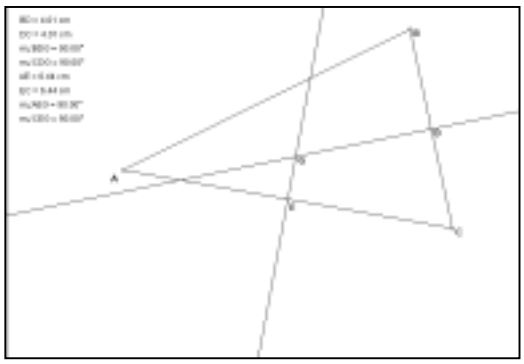


Figure 8

3. Invite group representatives to report their findings. The conditions for constructing the lines can be concluded. Then the teacher may introduce the names of these special lines, namely medians, altitudes, angle bisectors and perpendicular bisectors respectively for Figures 1 to 4.

4. Students are given another *Sketchpad* file tri5.gsp as shown in Figure 9. The medians (L_1, L_{11}), angle bisectors (L_2, L_{22}), altitudes (L_3, L_{33}) and perpendicular bisectors (L_4, L_{44}) of ΔABC can be shown or hidden by pressing the **Hide/Show** buttons (See Figure 10).

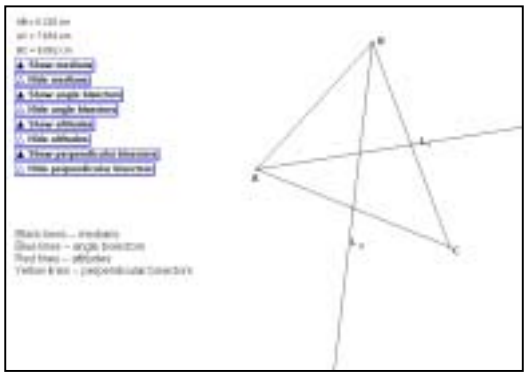


Figure 9

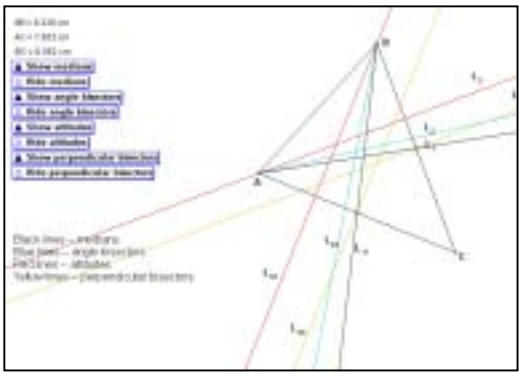


Figure 10

5. Through dragging the ΔABC, students are asked to answer the following questions in the Section B of the Worksheet:

   (a) Do all the lines pass through the opposite vertex in any shapes of ΔABC?

   (b) What types of triangle will make some of the lines L_1 to L_{44} merge into one?

   (c) What types of triangle will make the four lines L_1, L_2, L_3 and L_4 and the four lines L_{11}, L_{22}, L_{33} and L_{44} become one line respectively?

   (d) Whether the intersecting points of (i) medians, (ii) altitudes, (iii) angle bisectors, (iv) perpendicular bisectors always lie inside the triangle? Under what circumstances the intersecting points will lie outside the triangle?

6. Some students are invited to present their findings. The teacher can then draw the conclusion and demonstrate the situation in point 5(c) above for the case of equilateral triangle if deemed necessary. Further explanation of the condition of congruence of triangles can be highlighted if students show interest in the reasons behind.

Worksheet 1.1A Investigate the lines in a triangle

Section A

1. Investigate the lines AD and BE in the *Sketchpad* files tri1.gsp. Describe how these 2 lines can be constructed.

   _____

2. Investigate the lines AD and BE in the *Sketchpad* files tri2.gsp. Describe how these 2 lines can be constructed.

   _____

3. Investigate the lines AD and BE in the *Sketchpad* files tri3.gsp. Describe how these 2 lines can be constructed.

   _____

4. Investigate the lines GD and GE in the *Sketchpad* files tri4.gsp. Describe how these 2 lines can be constructed.

   _____

Section B

Regarding medians, altitudes, angle bisectors and perpendicular bisectors of ΔABC in the file tri5.gsp, answer the following questions:

5. Do all the lines pass through the opposite vertex in any shapes of ΔABC?

   _____

6. What types of triangle will make **some** of the lines L_1, L_{11}, L_2, L_{22}, L_3, L_{33}, L_4 and L_{44} merge into one? Show your sketch on the screen. In this particular triangle, do all 2 lines of the same type (say, L_1 and L_{11}) always merge with the 2 lines of the other types (say, L_2 and L_{22})? Why or why not?

   _____

7. What type(s) of triangle will make the four lines L_1, L_2, L_3 and L_4 and the four lines L_{11}, L_{22}, L_{33} and L_{44} coincide respectively?

   _____

8. Whether the intersecting points of (i) medians, (ii) altitudes, (iii) angle bisectors, (iv) perpendicular bisectors always lie inside the triangle? Why?

   _____

Worksheet 1.1B Investigate the lines in a triangle

Section A

1. Investigate the lines AD and BE in *Sketchpad* file tri1.gsp. For the following items, put a tick if it is true.

   ☐ Each line passes through one of the vertices of the triangle.
   ☐ Each line bisects one of the sides of the triangle
   ☐ Each line bisects one of the interior angles of the triangle.
   ☐ Each line is perpendicular to one of the sides of the triangle

   Describe briefly how to construct these lines.

   _____

   _____

2. Investigate the two lines AD and BE in *Sketchpad* file tri2.gsp. For the following items, put a tick if it is true.

   ☐ Each line passes through one of the vertices of the triangle.
   ☐ Each line bisects one of the sides of the triangle
   ☐ Each line bisects one of the interior angles of the triangle.
   ☐ Each line is perpendicular to one of the sides of the triangle

   Describe briefly how to construct these lines.

   _____

   _____

3. Investigate the two lines AD and BE in *Sketchpad* file tri3.gsp. For the following items, put a tick if it is true.

   ☐ Each line passes through one of the vertices of the triangle.
   ☐ Each line bisects one of the sides of the triangle
   ☐ Each line bisects one of the interior angles of the triangle.
   ☐ Each line is perpendicular to one of the sides of the triangle

   Describe briefly how to construct these lines.

   _____

   _____

4. Investigate the two lines GD and GE in *Sketchpad* file tri4.gsp. For the following items, put a tick if it is true.

   ☐ Each line passes through one of the vertices of the triangle.
   ☐ Each line bisects one of the sides of the triangle
   ☐ Each line bisects one of the interior angles of the triangle.
   ☐ Each line is perpendicular to one of the sides of the triangle

   Describe briefly how to construct these lines.

   _____

   _____

Section B

By dragging the vertices of the ΔABC to explore the properties of lines L_1, L_{11}, L_2, L_{22}, L_3, L_{33}, L_4 and L_{44} in the *Sketchpad* file tri5.gsp:

5. Do all the lines pass through the opposite vertex in any shapes of ΔABC? Which one does not?

   _____

6. What types of triangle will make **some of the lines** L_1, L_{11}, L_2, L_{22}, L_3, L_{33}, L_4 and L_{44} merge into one, for example, L_1 and L_2? Show your sketch on the screen. In this particular triangle, do all 2 medians merge with all 2 angle bisectors?

   _____

7. What type(s) of triangle will make the four lines L_1, L_2, L_3 and L_4 and the four lines L_{11}, L_{22}, L_{33} and L_{44} coincide respectively?

   _____

8. Whether the intersecting points of (i) medians, (ii) altitudes, (iii) angle bisectors, (iv) perpendicular bisectors always lie inside the triangle? Under what circumstances the intersecting points will lie outside the triangle?

   _____

   _____

Part B：To explore the properties related to the medians of a triangle

1. Distribute Worksheet 1.2 to students. Students are asked to construct the medians in a triangle by following the instructions of Construction 1 in the Worksheet. The teacher can invite students who are more competent in using *Sketchpad* to help other students. Students can then use the figure to explore the concurrence of the medians.

2. Ask students to do Investigation 1 in the Worksheet:
   (a) When you construct the third median, does it pass through the point G?
   (b) Move the vertices of the triangle. Does the way the medians intersect change?
   (c) Write a conjecture about the way the medians intersect in a triangle.

3. After Investigation 1, the teacher can lead the class to discuss the way of intersection of medians in triangles. The concurrence of all medians and the name of the intersecting point should be concluded with the emphasis that this phenomenon is true for any triangle.

4. Ask students to make other conjectures to the properties of the centroid by dragging the vertices of the triangle. The teacher can ask questions focussing on the lengths if students do not have any idea. Time for students to make guesses should be allowed. After students have an intuitive idea on the ratio at which the centroid divides the medians, students are invited to share their ways of checking the conjectures. After then, students can follow Construction 2 in the lower part of the Worksheet and Investigation 2 of the Worksheet to check their conjectures:
   (a) Calculate the ratios $\frac{AG}{AD}, \frac{BG}{BE}$ and $\frac{CG}{CF}$. What do you notice about the ratios? Do these ratios change when you move the triangle?
   (b) Calculate the ratios $\frac{AG}{GD}, \frac{BG}{GE}$ and $\frac{CG}{GF}$. Make a conjecture about the way the centroid G that divides the medians of a triangle.

5. After Investigation 2, the teacher can ask students to conclude their findings. The teacher should help students to conclude the invariance of the ratio quantity for any type of triangle. The teacher should remind students that using measurement is only one way of counter-checking their conjectures and guide students to see the limitation of this method. For more able students, the teacher may further motivate students to think about the deductive proofs on the finding by working through Part C of the Activity.

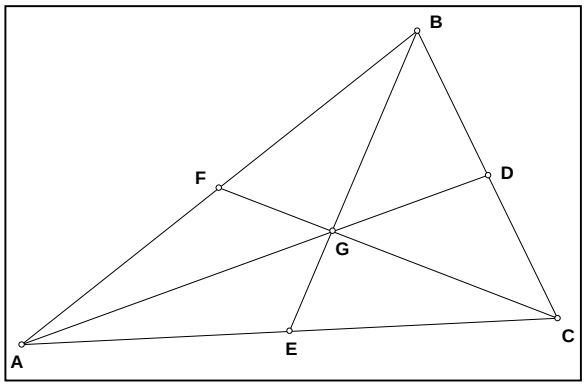
Worksheet 1.2 Explore the properties related to the medians of a triangle

You are asked to discover the way the medians of a triangle related to each other. The steps are as follows:

- Use *Sketchpad* and follow the instruction to construct the medians in a triangle.
- Use the sketch to investigate relationships involving medians.
- Write down your answers to the questions in the worksheet.

Construction 1：

(a) Draw a ΔABC.
(b) Construct the midpoint of segment BC.
(c) Label the midpoint as D.
(d) Draw a segment from vertex A to midpoint D.
(e) Construct the second median to side AC in the same manner. Label the midpoint of segment AC as E.
(f) Label the point of intersection of these two medians as G.



(g) Construct the third median to side AB in the same manner. Label the midpoint of segment AB as F.

Investigation 1：

1. When you construct the third median, does it pass through the point G?

   _____

2. Move the vertices of the triangle. Does the way the medians intersect change?

   _____

3. Write a conjecture about the way the medians intersect in a triangle.

   _____

Construction 2：

(a) Measure the length of the segment AG.

(b) Measure the length of the segment AD.

(c) Measure the length of the segment GD.

(d) Measure the lengths of other medians and their parts in the same manner.

Investigation 2：

1. Calculate the ratios $\frac{AG}{AD}, \frac{BG}{BE}$ and $\frac{CG}{CF}$. What do you notice about the ratios? Does this change when you move the triangle?

   _____

2. Calculate the ratios $\frac{AG}{GD}, \frac{BG}{GE}$ and $\frac{CG}{GF}$. Make a conjecture about the way the centroid G that divides the medians of a triangle.

   _____

Part C：To give formal geometric proofs

1. After the investigations in Part B, students should discover that the centroid divides the medians in the ratio 2:1 and the medians are concurrent for any types of triangles. The teacher guides students in laying out strategies to prove each of their discoveries. Various approaches may be discussed.

2. For average students, the teacher can ask them to use similarity and congruence of triangle for the proof (See Point 4 of the **Notes for Teachers**, Methods 1 & 2).

3. Some hints can be given to those students. For instance, for point 4 of Method 1 in the **Notes for Teachers**, the teacher can use the "working backwards" strategies in posing the following questions to guide students sorting out the strategies (see Figure 11 for reference):
   (a) If we want to prove AG : GD = 2:1, which pair of triangles should be considered? Is there sufficient condition to prove the conjecture with focus only on ΔAFG and ΔGDC? Should we add additional lines to the figure?
   (b) If we add a line DH parallel to AB, which pair of triangles should be considered in proving the ratio? As we want to prove that ΔDHG ~ ΔAFG in order to justify the ratio, what conditions do we need in proving the similarity of the triangles? Which other pair of triangles can be considered as a link?
   (c) If ΔCDH and ΔCBF are the link, what relation between these triangles can help us to relate the lengths concerned? What is the ratio of BF: DH? And what is the ratio of AF : HD?
   (d) Can we prove ΔDHG ~ ΔAFG? What is the ratio of AG : GD?

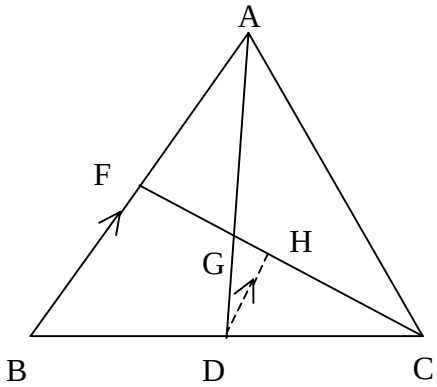


Figure 11

4. For more able students, the teacher can request them to use more than one method to carry out the proof (see **Notes for Teachers** point 4, Methods 1 to 4).

5. The more able students may try to complete the proofs by themselves. Nevertheless, the teacher should give some hints if they have problems.

Notes for Teachers：

1. The teacher can use the *Sketchpad* files tri11.gsp, tri22.gsp, tri33.gsp & tri44.gsp to demonstrate the figures 5 to 8. Further, the teacher can use the *Sketchpad* file tri5_2.gsp to demonstrate the merging of lines as in Figure 9.

2. Suggested answers to Worksheet 1.1：

   Section A

   1. Each line extends from one vertex of the triangle to the midpoint of its opposite side.
   2. Each line extends from one vertex of the triangle and perpendicular to the opposite side.
   3. Each line bisects the angle of the triangle.
   4. Each line is both perpendicular to the side of the triangle and passes through its midpoint.

   Section B

   5. No.
   6. For an isosceles triangle with AB = AC, only the lines L_1, L_2, L_3 and L_4 will merge into one. Similarly, if AB = BC, the lines L_{11}, L_{22},, L_{33}, and L_{44} will merge into one.
   7. Equilateral triangle.
   8. The intersecting point of altitudes and the intersecting point of perpendicular bisectors will lie outside the triangle if the triangle is an obtuse-angled triangle.

3. Suggested answers to Worksheet 1.2：

   Investigation 1

   1. Yes.
   2. No.
   3. The medians of a triangle intersect at a point.

   Investigation 2

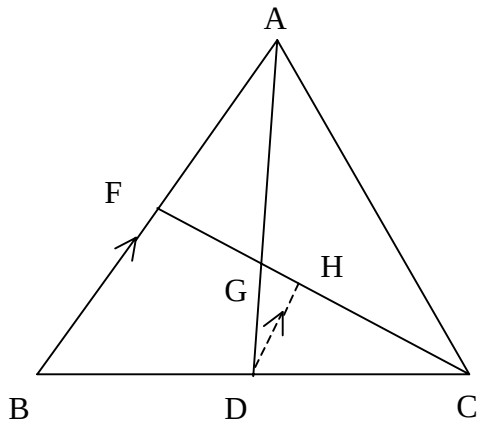
   1. $\frac{2}{3}$.
   2. These three ratios are 2 or $\frac{2}{1}$. In other words, we can say that the centroid divides the medians in a ratio of 2 to 1.

4. Suggested answers to Part C

(1) Prove that the centroid of a triangle divides the medians in the ratio 2:1.

Method 1

Prerequisites: Conditions for similar triangles and the properties of similar triangles

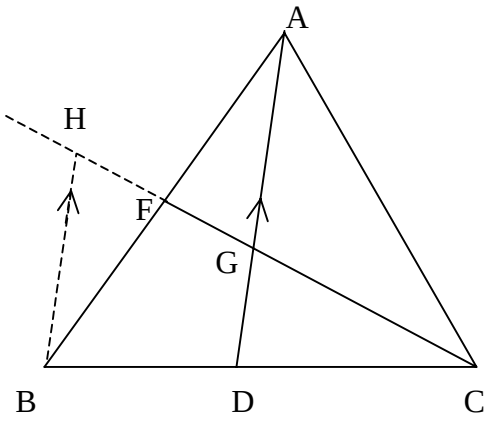


Key Procedures:

1. Construct medians AD and CF.
2. Construct DH // BA
3. Show that ΔCDH ~ ΔCBF
4. Consider the ΔCDH and ΔCBF, find the ratio BC : DC and AF: HD
5. Show that ΔDHG ~ ΔAFG and find the ratio AG : GD.

Method 2

Prerequisites：Conditions for similar and congruent triangles and the properties of similar triangles

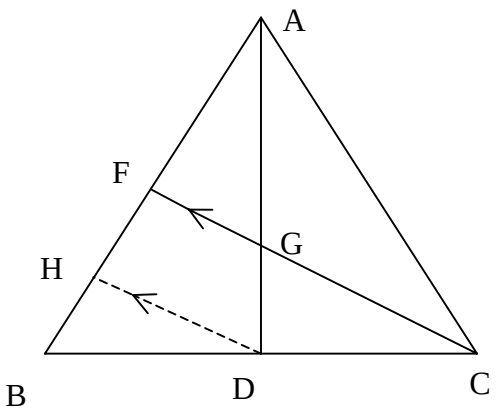


Key Procedures:

1. Construct medians AD and CF
2. Produce CF.
3. Construct BH // AD
4. Show that ΔBHF ≅ ΔAGF
5. Show that ΔCDG ~ ΔCBH
6. Find the ratio BH : DG and AG : GD.

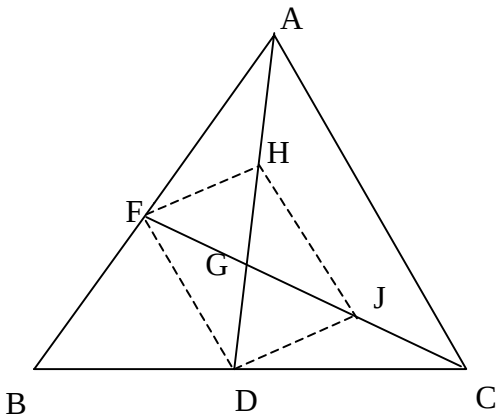
Method 3

Prerequisite：Intercept Theorem

	Key Procedures: 1. Construct medians AD and CF 2. Construct DH // CF 3. Consider the ΔBCF and ΔBDH, apply Intercept Theorem to show that BH = HF 4. Find the ratio AF : FH 5. Consider the ΔAHD and ΔAFG, apply Intercept Theorem again to get the ratio AG : GD.

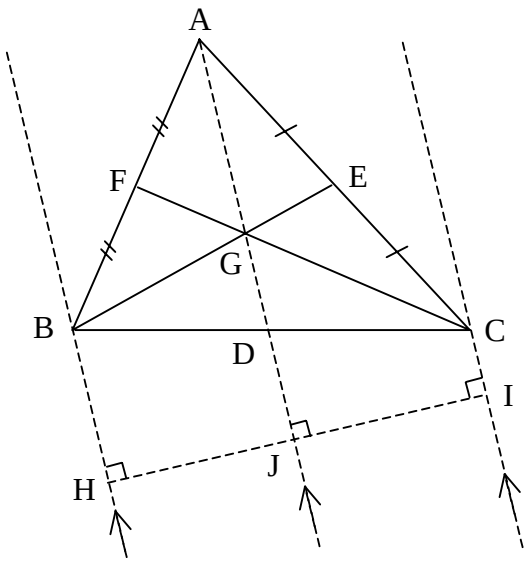
Method 4

Prerequisites：Mid-point Theorem and the properties of parallelogram.

	Key Procedures: 1. Construct medians AD and CF 2. Construct mid points H and J of AG and CG respectively and form quadrilateral DFHJ 3. Consider the ΔBFD, ΔBAC and then ΔGJH and ΔGCA, apply the Mid-point Theorem to show that JH = DF and JH // DF and hence deduce that DFHJ is a parallelogram 4. Use the properties of parallelogram to find the ratios FG : GJ, CG : GF and AG : GD

(2) Prove that the medians are concurrent.

Prerequisite：area of triangle, centroid of a triangle divides the medians in the ratio 2:1, ratio of areas of two triangles having same height = ratio of their base lengths.



Key Procedures:

1. Construct medians BE and CF
2. Construct line AD, which passes through the point of intersection of BE and CF
3. Construct the lines BH and CI such that AD//BH//CI
4. Construct line HI such that HI ⊥ AD
5. Show that the area of ΔAFG = the area of ΔBFG
6. Show that the area of ΔAEG = the area of ΔCEG
7. Apply the fact BG : GE = CG : GF = 2 : 1 to show the area of ΔAFG = the area of ΔAEG
8. Apply the result of 7 to show that HJ = JI
9. Show that the area of ΔBGD = the area of ΔCGD
10. Find the ratio BD : DC and hence show that AD is the median of the triangle

Reference：

Web Sites:

1. http://www.geom.umn.edu/~demo5337/Group2/trianglecenters.html
2. http://cedar.evansville.edu/~ck6/tcenters/index.html
3 http://aleph0.clarku.edu/~djoyce/java/Geometry/eulerline.html

Books:

1. Perham, Arnold E., Perham, Bernadette H., Perham, Faustine L. (1997). “Creating a Learning Environment for Geometric Reasoning”. *Mathematics Teachers*, 90(7) pp.521 – 526. Reston, Virginia: National Council of Teachers of Mathematics.

2. 中國教育學會主辦。《中小學數學—初中版》。1999 年第 7 – 8 期 (頁八)。

3. Battista, Michael T. (1998). *Shape makers: developing geometric reasoning with the Geometer’s Sketchpad*. Emeryville, California: Key Curriculum Press.

4. Dixon, Robert A. (1991). *Mathographics*. New York: Dover Publications.

5. Wyatt, Karen W., Lawrence, Ann and Foletta, Gina M. (1998). *Geometric activities for middle school students: with the Geometer’s Sketchpad*. Emeryville, California: Key Curriculum Press.

6. Yerushalmy, Michal and Houde, Richard (1987). *Geometry problems and projects: triangles*. Pleasantville, New York: Sunburst Communications.