Exemplar 19:
Applying the Idea of Transformation in Geometric Proofs

Objective: To apply the idea of transformation in doing geometric proofs

Key Stage: 3

Learning Unit: Quadrilaterals

Materials Required: Nil

Prerequisite Knowledge:
(1) Basic understanding of the conditions for congruence
(2) Basic idea of transforming a figure
(3) Basic idea of properties of squares

Description of the Activity:

1. The teacher reminds students the general steps in doing geometric proofs: To distinguish the given conditions and the proof requested.

2. Worksheet 1 with the following problem is given to students:

In Fig. 1, $ABCD$ is a square. Points E and F are respectively on AB and CB. Given that $\angle ADE = \angle EDF$, prove that $AE + CF = DF$.

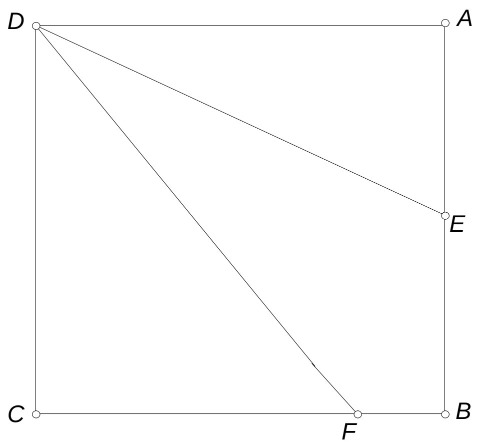


Fig. 1

3. The teacher asks students to study the problem and to identify:
   (a) the given conditions of the problems;
   (b) what is needed to be proved.

4. Students are asked to solve the problem. Discussion may then be held with students on the strategies they used in solving the problem if they can solve it by their own. Otherwise, the teacher guides students to use the idea of transformation in solving the problem.

5. The following questions can be raised:
   (a) In Fig. 1, are the given conditions and the proof requested related together?
   (b) Can we link the 2 line segments *AE*, *CF* together?
   (c) Can we transform some triangles to link the 2 line segments together? Which triangles can be used?
   (d) If we consider Δ*ADE* and Δ*DCF*, can we do "something" to "combine" the 2 triangles into one? How can we do that?

   Through discussion, the teacher guides students to come up to the strategies of rotating Δ*ADE* by 90° around point *D* in the clockwise direction (Fig. 2).

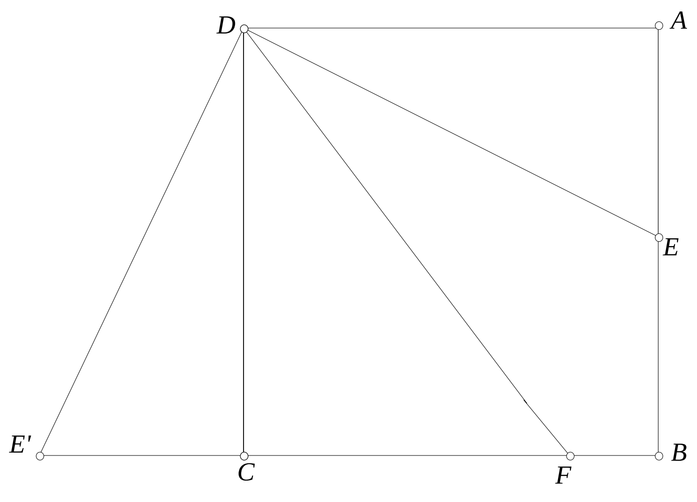


Fig. 2

6. After this transformation, the teacher asks students whether Δ*DCE'* and the Δ*DAE* are the same (or congruent) and why. Discussion should be made on whether the *E'CF* is a straight line. Students are then asked what changes can be seen on the proof requested "*AE* + *CF* = *DF*".

7. The teacher can ask the average or more able students to complete the proof in the worksheet. But for less able students, the teachers can raise the following questions to help students to complete the solution step by step:
   (a) As *AE* + *CF* becomes *E'C* + *CF*, that is *E'F*, which triangle can be used to link up the *E'F* and *DF* together?
   (b) In considering Δ*DE'F*, if we want to prove that the 2 sides of the triangle are the same, what should we do?

8. After solving the problem in Worksheet 1, the teacher may ask students to solve a similar problem in Worksheet 2.

9. Students are invited to present their solutions. The teacher then summarizes the steps used.

10. The teacher may illustrate another figure (Fig. 3) and guides students to apply other type of transformation - translation to perform the proof. The problem is:

    In Fig. 3, *AB* = *AC*. *D* is a point on *AB* and *E* is a point on the prolonged line segment *AC* with *DB* = *CE*. Prove that *DE* > *BC*.

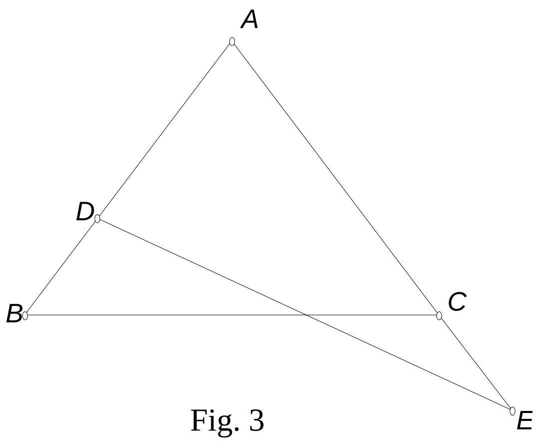


Fig. 3

11. Worksheet 3 is given to students. Discussion is made on the strategies to solve the problems. Students are then asked to solve the problem in the given space. After then, (Refer **Notes for Teachers** for the solution). The teacher can discuss with students the advantage of using the idea of transformation in doing geometric proofs

12. For more able students, the teacher can provide another 2 problems for them as enrichment activity or as homework assignment. The problem is as follows:

In Fig. 4, *ABCD* is a square. Given that *E* is a point inside the square such that $\angle ECB = \angle EBC = 15°$. Using transformation technique, prove that ΔADE is an equilateral triangle.

(Hint: add *O*, the center of the square, to the diagram.)

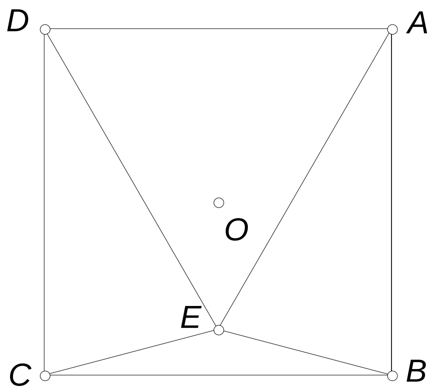


Fig. 4

In Fig. 5, ΔABC is an isosceles triangle with $\angle A = 100°$. Construct an angle bisector of $\angle B$ and intersect the line *AC* at the point *D*. Prove that $AD + BD = BC$.

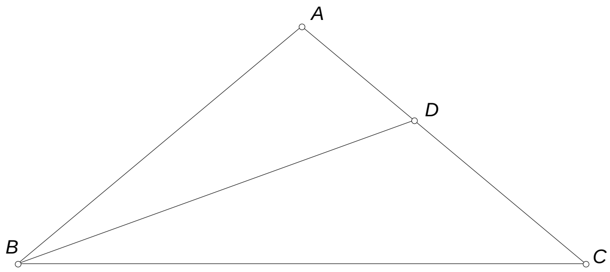
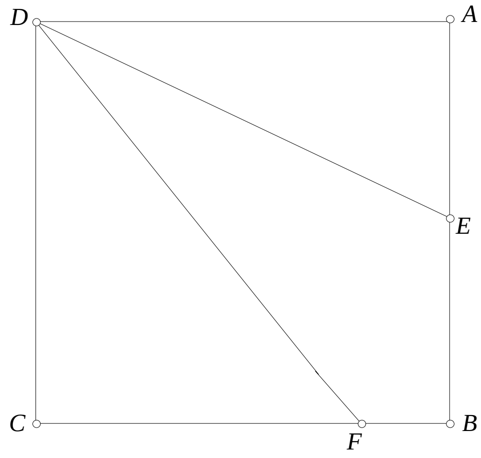


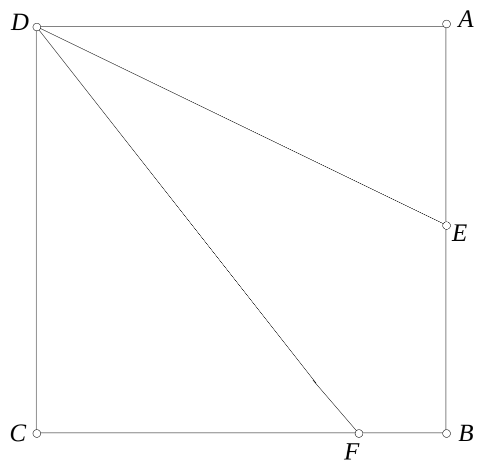
Fig. 5

Worksheet 1: Geometric Proofs using the Idea of Rotation

In the figure, *ABCD* is a square. Points *E* and *F* are respectively on *AB* and *CB*. Given that $\angle ADE = \angle EDF$, prove that $AE + CF = DF$.

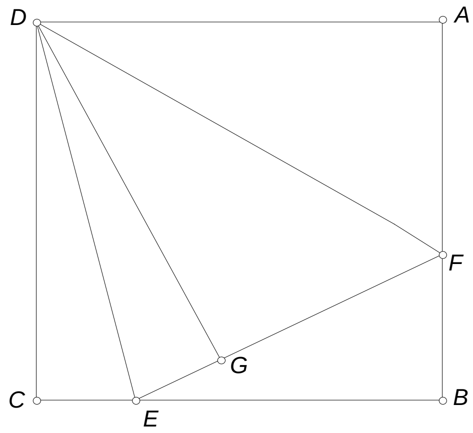


Proof:

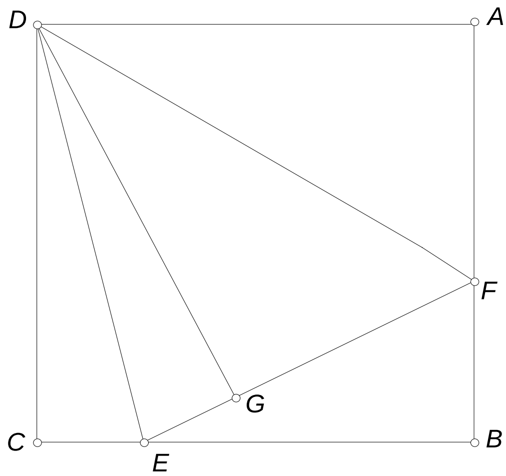


Worksheet 2: Geometric Proofs Using the Idea of Rotation

In the figure, *ABCD* is a square. Points *E* and F are respectively on *CB* and *AB*. Given that $\angle FDE = 45°$ and $DG \perp EF$, prove that $DG = DC$.

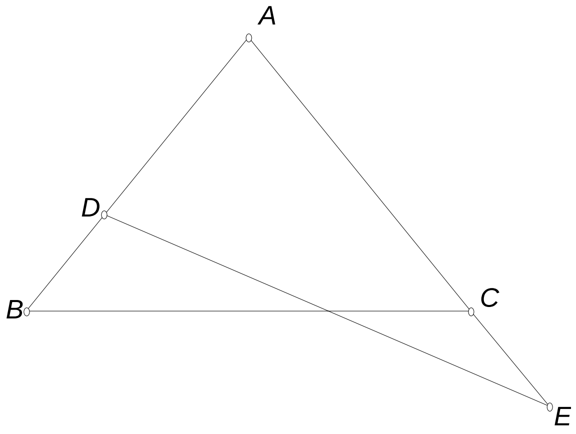


Proof:

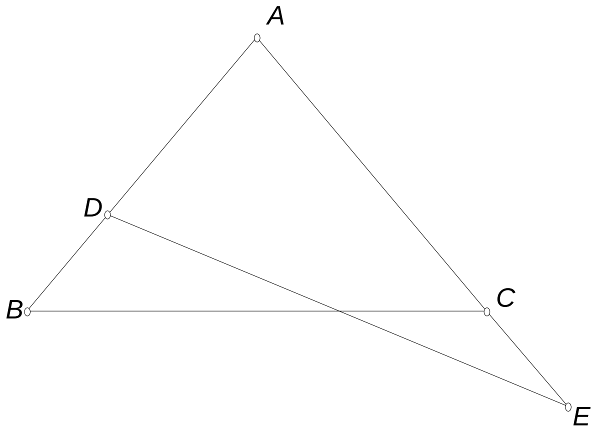


Worksheet 3: Geometric Proofs Using the Idea of Translation

In the figure, $AB = AC$. D is a point on AB and E is a point on the prolonged line segment AC with $DB = CE$. Prove that $DE > BC$.



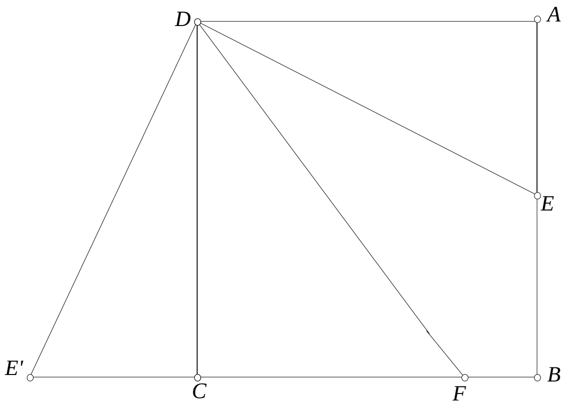
Proof:



Notes for Teachers:

1. The idea of symmetry and transformation is a new topic in the *Syllabus*. Although students may find it not difficult to study the topic, it is not easy for students to integrate this idea into other geometric problems. In solving geometric problems, students always consider the figure as static and they are not flexible enough to add lines/figures in building up linkage in solving non-routine problems. This exemplar illustrates how students may transform the geometric figure to build up a linkage in analysing and solving geometric problems.

2. Solution for Worksheet 1 is suggested as follow:



To prove: $AE + CF = DF$.

Key Procedures:

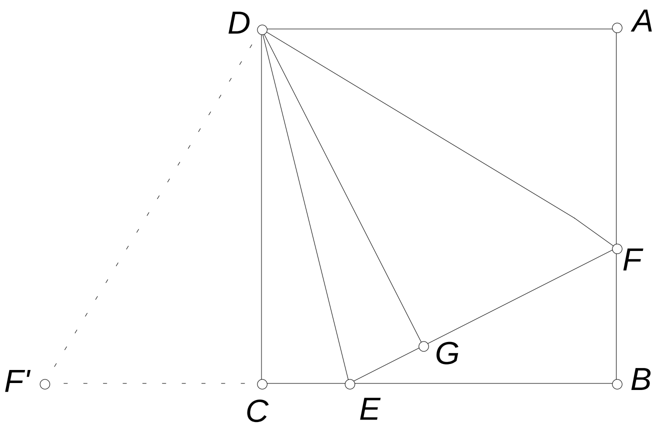
1. Rotate ΔADE by 90° around point D in the clockwise direction.
2. Show that $\Delta ADE \cong \Delta CDE'$ and $E'CF$ is a straight line.
3. Show that $AE = E'C$ and $AE + CF = E'F$.
4. Show that $\angle DE'F = \angle FDE'$.
5. Thus, show that $E'F = DF$ and so as the requirement of the problem.

Instead of rotating ΔADE, ΔDCF can be rotated to form $\Delta DAF'$. The proof is similar to that above.

3. In guiding students to solve the problem, it is very important to raise questions to link up the conditions and the proof requested. Time should be allowed for students to sort out different possible strategies instead of just giving out the answer right after students cannot devise the plan at the first moment. Nevertheless, the teacher may guide students to focus the strategies in using transformation techniques.

4. In transforming ΔADE to form $\Delta CDE'$, the focus is then changed to $\Delta DE'F$. This strategy of converting the problem into another problem can be highlighted as one of the important problem solving strategies.

5. Solution for Worksheet 2 is as follow:

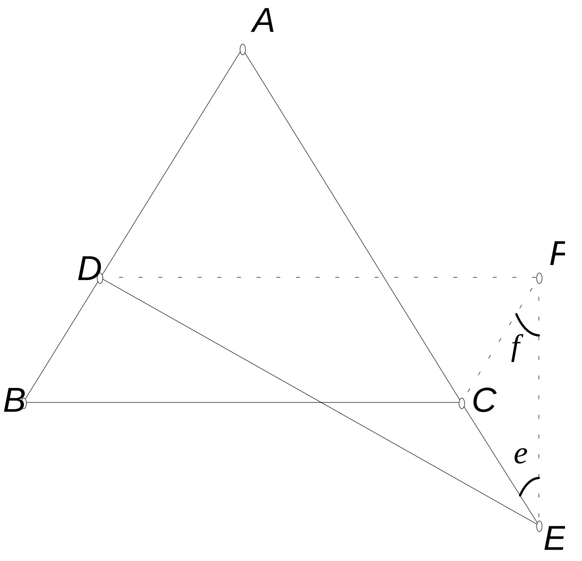


To prove: *DG* = *DC*.

Key Procedures:

1. Show that $\angle CDE + \angle FDA = 45°$.
2. Rotate ΔADF by 90° around point D in the clockwise direction.
3. Show that $\Delta DEF \cong \Delta DEF'$.
4. Thus, show that *DG* = *DC*.

6. It should be noted that the problem for Worksheet 3 requires a basic understanding of the laws of inequalities of triangles. Solution for Worksheet 3 is as follow:

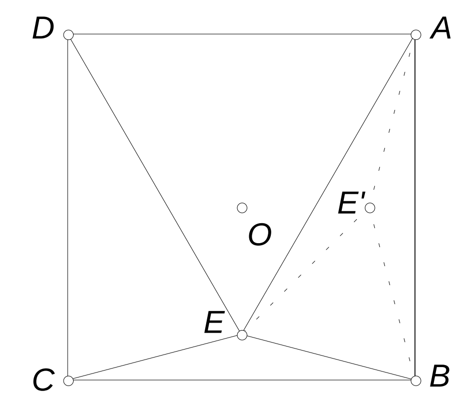


To prove: *DE* > *BC*.

Key Procedures:

1. Translate *BC* along *BD* to form *DF* and link *CF*.
2. Hence, *BCFD* is a parallelogram, show that *CF* = *CE*.
3. Link *EF* and show that *f* = *e*.
4. Show that $\angle DFE > \angle DEF$, hence, show that *DE* > *BC*.

7. The key procedures in solving the problem stated in Fig. 4 of the Description of the Activity are shown as below:



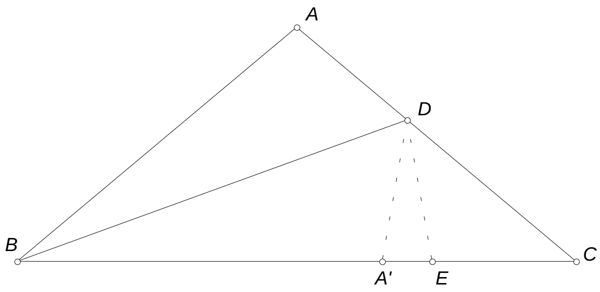
To prove: ΔDEA is an equilateral triangle.

Key Procedures:

1. Rotate ΔEBC around point O in anti-clockwise 90° to form $\Delta AE'B$.
2. Show that $EB = E'B$.
3. Show that $\Delta E'BE$ is an equilateral triangle.
4. Show that $E'E = E'A$.
5. Find $\angle AE'E$ and $\angle E'AE$.
6. Show that $\angle EAD = 60°$.
7. Similarly, show that $\angle EDA = 60°$.
8. Find $\angle DEA$.
9. Hence, show that ΔDEA is an equilateral triangle.

This problem is also a good example to illustrate how different strategies can be used to solve problems. Solution can be found in the article "Different Methods to solve a Plane Geometry Question" in *EduMath* 9 published by the Hong Kong Association for Mathematics Education.

8. The key procedures in solving the problem in Fig. 5 of the Description of the Activity are as follows:



To prove: $AD + BD =$ BC.

To make AD, BD and BC in the same line segment before proving to prove the required statement.

Proof:

1. Reflect point A about the line BD to the point A' on the line BC, show that $AD = A'D$
2. Rotate BD around point B until D falls on the line BC with $BE = BD$
3. Show that ΔDEC is an isosceles triangle
4. Show that $DE = DA'$, hence show that $AD + BD =$ BC.

Reference：

1. Chung, C.M. & Chiu, W.M. (1999). Different Method to solve a Plane Geometry Question. In *EduMath* 9. Hong Kong: Hong Kong Association for Mathematics Education.

2. 蘇和平(2000)。旋轉變換在平面幾何中的應用。《數學教學研究》。2000 年第 6 期。中國：西北師範大學。

3. 曹翔業(1997)。談認記圖形的教學。《中小學數學》(初中版)。1997 年第 12 期。中國：中國教育學會。

4. 初中數學奧林匹克刊授講座(初二) — 平移、對稱與旋轉(2000)。《中小學數學》(初中版)。2000 年第 6 期。中國：中國教育學會。