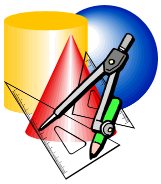


Exemplar 21:

The Ancient Chinese Proofs on Pythagoras' Theorem

Objectives: (1) To recognize the development of Pythagoras' Theorem in Ancient China
(2) To appreciate the contribution of Chinese in the development of mathematical knowledge

Key Stage: 3

Learning Unit: Pythagoras' Theorem

Materials Required: (1) Worksheets developed from related books
(2) Activity materials on the proofs by Liu Hui[1]

Prerequisite Knowledge: Basic understanding of Pythagoras' Theorem

Description of the Activity:

1. The teacher asks students to state Pythagoras' Theorem.

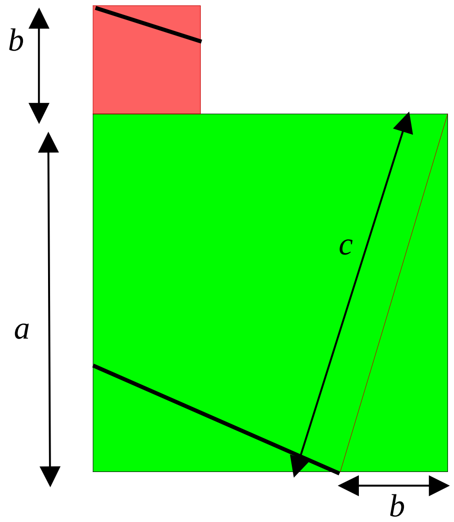
2. Worksheet 1 is given to students. They are asked to
   - extract as many equivalent names of Pythagoras' Theorem as possible;
   - find the number of proofs of Pythagoras' Theorem.

3. The teacher summarizes the names extracted: the *Gou-gu* Theorem[2] and *Shang-gao* Theorem[3]. The teacher may also remind students that the *Shang-gao* Theorem is used in some reference books especially those published in Taiwan. The teacher briefly explains the origin of the name *Gou-gu* Theorem and asks students to debate about the names of the Theorem. The parts (a) to (c) of question 3 can be assigned as homework assignment.

[1] The Chinese name of Liu Hui is 劉徽.
[2] The Chinese name of *Gou-gu Theorem* is 勾股定理.
[3] The Chinese name of *Shang-gao* Theorem is 商高定理.

4. The teacher distributes Annex and Worksheet 2 to students. The teacher briefly explains the proof used by Zhao Shuang[4] (Refer to **Annex** or **Notes for Teachers** for another similar strategy). The teacher guides students to observe the beauty of this proof. They are then asked to complete the proof by their own. Finally, the teacher concludes the solution of the proof.

5. The teacher then introduces the name and the background stories of another Ancient mathematician Liu Hui[5]. Students are given the prepared materials in the following layout. They are asked to use the least steps to move the 5 pieces to form a large square with a side of c.



6. Some students are invited to demonstrate the steps and discussion on the method of the least steps is held. The teacher then distributes Worksheet 3 and introduces Liu Hui's method and his diagram.

7. The teacher summarizes the two proofs used by the Ancient Chinese mathematicians and compares the proof written in other countries such as Mr. Garfield's proof, the former USA President. The teacher then guides students to appreciate the contribution of Chinese in the development of mathematical knowledge. For students who are interested in the topic, the teacher may give Worksheet 4 to them to explore the application of the *Gou-gu* Theorem in ancient China.

[4] The Chinese name of Zhao Shuang is 趙爽.

[5] Liu Hui is famous in his method to find the approximate value of π. His method is to dissect a circle into a large number of sides of regular polygons and use the perimeter of these polygons to approximate the circumference of the circle and hence the value of π. The method is called exhaustion method or in Chinese 割圓術.

Worksheet 1: Names of Pythagoras' Theorem

Read the following paragraph and answer the following questions.

There is a very famous theorem in Geometry – Pythagoras' Theorem. Pythagoras was a Greek philosopher, astronomer, mathematician and musician in around 500BC. Although, it is widely accepted that the Theorem is named as Pythagoras' Theorem (or Pythagorean Theorem), there are still lots of arguments saying that other mathematicians earlier than Pythagoras found this Theorem. In the Mainland, the Theorem is called *Gou-gu*[1] Theorem or *Shang-gao*[2] Theorem in Taiwan. The names *Gou* and *Gu* refer to the shorter sides of the right-angled triangle (*Yuan* is the hypotenuse of the triangle) whereas *Shang-gao* refers to a person in Zhou Dynasty (around 1100BC). Both names are found in the famous Chinese book *Zhou Bi Suan Jing*[3].

Zhou Bi Suan Jing is one of the oldest books of astronomy. The first chapter of it recorded a conversation between Duke Zhou and Shang-gao. Shang-gao's answers included a statement about a particular case of *Gou-gu* Theorem - "句[4]廣三， 股修四， 徑隅五". In English, it means "if the 2 shortest sides of a right-angled triangle are 3 and 4, the hypotenuse is 5. There was another conversation made by Rong Fang and Chen Zi[5] recorded in the book about the general form of the *Gou-gu* Theorem, that is $a^2 + b^2 = c^2$.

Although *Zhou Bi Suan Jing* was completed in years between 100BC and 100AD[6], it is believed that the contents of the book might probably appear much earlier that its completion, such as 1100AD[7]. Thus, there have been arguments on whether Pythagoras' Theorem should be renamed as *Gou-gu* Theorem.

Gou-gu Theorem is not only one of the oldest theorems; it is also a theorem with many different proofs. In the past thousand years, there are more than 400 proofs of the Theorem.

1. (a) Name as many equivalent names of Pythagoras' Theorem as possible.

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(b) How many proofs of Pythagoras' Theorem have been developed?

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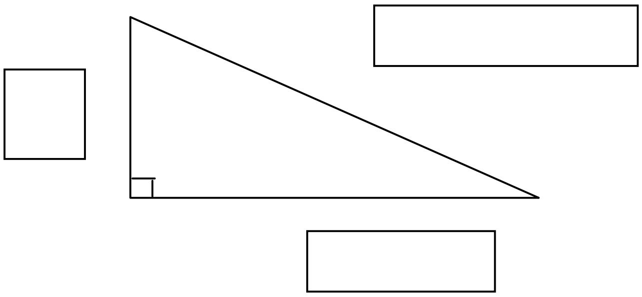
[1] The Chinese name of *Gou-gu* Theorem is 勾股定理.
[2] The Chinese name of *Shang-gao* Theorem is 商高定理. Shang-gao is believed as a descendant of Huang Di and he was very good at mathematics.
[3] The Chinese name of *Zhou Bi Suan Jing is* 周髀算經.
[4] "句" is the same as the present word "勾".
[5] The Chinese name of Rong Fang and Chen Zi are respectively 榮方 and 陳子.
[6] 李儼 (1992), P.31.
[7] 曲安京(1996).

2. Write the name "*Gou*", "G*u*", "*Yuan*"[8] in the figure below.



3. (a) What is the approximate year for the "discovery" of the Pythagoras' Theorem in Greek?

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(b) What is the approximate year for the completion of the *Zhou Bi Suan Jing?* Why is there argument for the year of discovery of *Gou-gu* Theorem?

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(c) Explain briefly why there is saying that the Pythagoras' Theorem should be named as *Gou-gu* Theorem.

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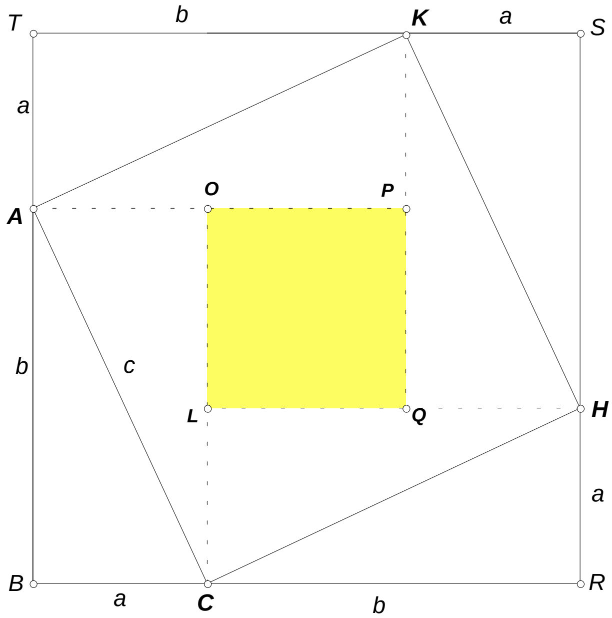
[8] The Chinese names of *Gou*, *Gu* and *Yuan* are 勾, 股, 弦 respectively.

Worksheet 2: Proof by Zhao Shuang[1]

1. Read the article in Annex about Zhao Shuang (around 300AD) in his commentaries of *Zhou Bi Suan Jing*[2] on the *Gou-go* Theorem. Refer to the proof written in the articles. Rewrite the complete proof in mathematical forms and fill in the below box:



Solution:

[1] The Chinese name of Zhao Shuang is 趙爽.
[2] The Chinese name of *Zhou Bi Suan Jing* is《周髀算經》.

Worksheet 3: Proofs by Liu Hui[1]

In the contemporary of Zhao Shuang, a Chinese mathematician Liu Hui found another wonderful proof of *Gou-gu* Theorem. In Liu's proof, he could prove the Theorem without using algebraic method. The illustration for his method (in Chinese) is as follows:

> 句[2]自乘為朱方，股自乘為青方，令出入相補，各從其類，因就其餘不移動也。合成弦方之冪，開方除之，即弦也。

Method used by Liu Hui :

Liu Hui first defined the adjacent and opposite sides of the right-angled triangle to construct two squares. He named the two squares formed as the "red square" and the "green square" (Fig. 1). Then he marked "cut" and "paste" in the figure. He moved the cut portion to the paste portion correspondingly. It thus formed the tilted square (Fig 2).

Liu Hui named the titled square as "Yuan square" and it is the square formed by the hypotenuse of the original right-angled triangle. After a series of cutting, translating and pasting, it will naturally come to the solution that

The (area of) red square + the (area of) green square = the (area of) Yuan square

i.e. $a^2 + b^2 = c^2$.

The Principle of Congruence by Subtraction and Addition"[3] is so wonderful that the proof can be easily understood without using any words.

Translated from《數學奇觀》 P. 61

[1] The Chinese characters for Liu Hui are 劉徽.
[2] "句" is the same as the present word "勾".
[3] This method is called 出入相補方法.

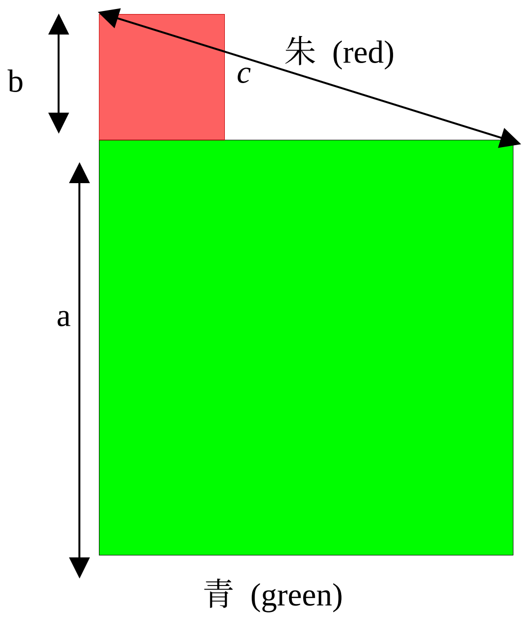


Fig. 1

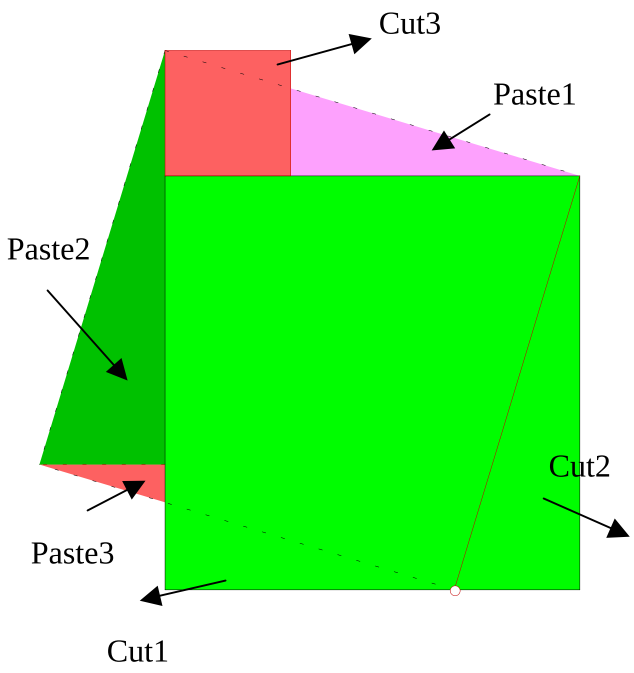


Fig. 2

Worksheet 4: The famous Reed (Lotus) problem

1.

The famous reed problem found in the *Gou-gu* (Chapter 9) of *Nine Chapters of Mathematical Art*[1] is as follow: “ 葭生中央問題 ” 今有池方一丈，葭生其中央，出水一尺，引葭赴岸，適與岸齊，問水深葭長各幾何? The meaning in current Chinese is: 有一個正方形的池塘，邊長為 1 丈，有棵蘆葦生長在池塘的正中央，高出水面的部分有 1 尺長，如果把蘆葦向岸邊拉，葦頂正好能碰到池岸邊沿。問池塘水深和蘆葦的長度各是多少？	The English translation for the poem is as follow: 1. In a square pond with a side of 10 feet[2], a reed is grown in the middle of the pond and is 1 foot above the water level. 2. If the reed is pulled to the bank of the pond, the tip of the reed just touches the bank. 3. Find the depth of the pond and the length of the sea reed.

(a) Draw the problem in diagram and solve the problem:

Solution:

[1] The Chinese name of *Nine Chapters of Mathematical Art* is 九章算術.

[2] The unit used is different from the Chinese poem. “Feet” is used to simplify calculation and with a comparable length corresponding to the original poem.

(b)

The solution proposed in the Chapter is:	The English translation for the solution is:
把池塘邊長的一半自乘，再把蘆葦出水的那部分自乘，然後相減，將所得的差除以出水數的 2 倍，就是池塘的水深，加上出水數，就是蘆葦的長度。	1. Square the half length of the pond side 2. Square the length of the reed which is above water 3. Find the difference between pt. 1 and pt. 2 4. Divide the difference by 2 and then get the depth of the water level. 5. The sum of the length of the reed above water and the depth of the water level is the length of the reed.

Explain why this method can find the solution.

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2. Below is the famous lotus problem written by the Indian mathematician Blaskara Acharya (1114-1185AD). Use the similar method to solve the problem:

The translated Chinese version is as follow:	The translated English version is as follow:
平平湖水清可鑒，面上半尺生紅蓬； 出泥不染亭亭立，忽被強風吹一邊； 漁人觀看忙向前，花離原位兩尺遠； 能算諸君請解題，湖水如何知深淺。[3]	In a certain pond, the tip of a bud of a lotus was seen 0.5 feet above the surface of the water. The lotus was forced by the wind and gradually submerged at a distance of 2 feet away from the original position. Find the depth of the pond.

Solution:

[3] Most of the Chinese reading materials in this worksheet are extracted from《數學奇觀》and 中國古代數學簡史》published by 九章出版社. Thanks are given to Mr Suen Man-sin for granting us to use the materials in this learning package.

Notes for Teachers:

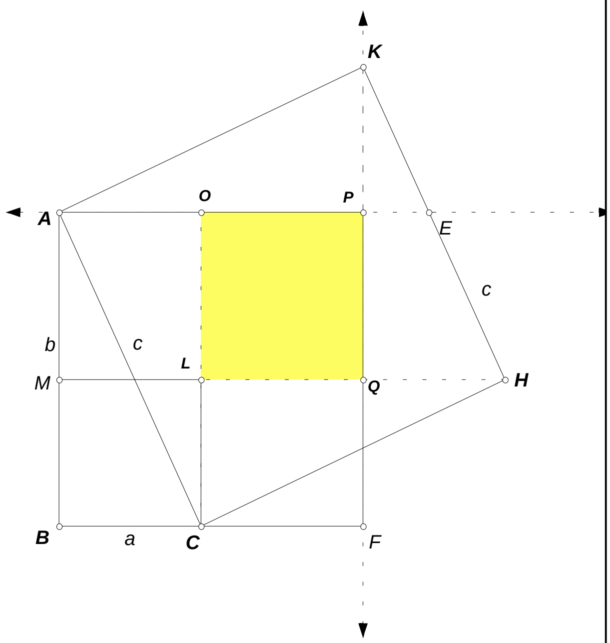
1. As this learning activity concerns with the proofs of Pythagoras' Theorem found in Ancient China, most reading materials are in Chinese. The English translation version may distort the original beauty of the language and the methods used. If possible, the teacher can give the original Chinese articles to students. Most articles or stories can be found in the reference books or websites provided.

2. The reading materials provided are just for reference. The teacher can replace any materials that found useful for discussion. The teacher is suggested to ask students to read some reading materials and conduct the discussion in class rather than just giving the materials as homework. The teacher can provide some other reading materials for students to read at home or even carry out a project on the topic.

3. Some Chinese reading materials are written in the language that are not easily to be understood. The teacher may just show the materials and then let students to work on the version written in the modern language. For students who are interested in understanding the old Chinese Language, the teachers may refer them to the Chinese teachers or jointly works with the Chinese subject as a cross-curricular activity.

4. In carrying out activities related to the history of mathematical knowledge, it is important to let students understand the dynamic nature of mathematical knowledge. It should also be noted that there are discrepancies between the historical details written in different books. The teacher should be very careful to handle this and select reading materials with reliable source for students. It is important to note that this activity focuses mainly on the contribution of Chinese on the development of the Theorem, but the teacher can also point out that other countries like Babylon had similar findings in their past records.

5. The teacher may use the idea of rotation in transformation to prove the diagram provided by Zhao Shuang:



Steps:

1. From the given ΔABC, construct squares *ACHK*, *BCLM* and *BFPA* with the side *c*, *a* & *b* respectively.
2. Rotate ΔABC by 90° around *C* in the clockwise direction to overlap ΔHLC.
3. Rotate ΔABC by 90° around *A* in the anti-clockwise direction to overlap ΔAPK.
4. *KPQ* is a straight line. Prolong the line *CL* to meet *AP* to form the rectangle *OLQP*.
5. As $OP = b-a = OL$, the rectangle *OLQP* is a square with the side as $b-a$
6. As $\Delta LCH \cong \Delta QHK \cong \Delta PKA \cong \Delta OCA$, $\therefore S_{ACHK} = 4S_{LCH} + S_{OLQP}$
7. Simplifying the above to get $c^2 = a^2 + b^2$.

6. The animation program for Liu Hui's proof can be found in the web-site http://www.math.ncu.edu.tw/~shann/Teach/calcware/3.15.html.

7. Worksheet 4 is designed to enrich students' understanding of the application of the *Gou-gu* Theorem in ancient China. The solution for the problems can be found in para. 6.6. of the book 《數學奇觀》. Another important application is to find the distance from the sun to the earth. This activity can be found in the web-page http://www.cmi.hk/Teaching/Pytha or in the book《中國古代數學簡史》.

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1. Darko Veljan (2000). The 2500-Year Old Pythagorean Theorem. In *Mathematics Magazine* Vol. 73, No.4, October 2000. Washington, DC: The Mathematical Assocation of America.
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3. 李學數 (1978) 。 "希臘郵票上的數學定理和中國的「商高定理」"《數學和數學家的故事(1)》。頁 1-9。香港：廣角鏡出版社。
4. 曲安京 (1996)。 "商高、趙爽與劉徽關於勾股定理的證明"《數學傳播》 20 卷 3 期。台北：中央研究院數學研究所。

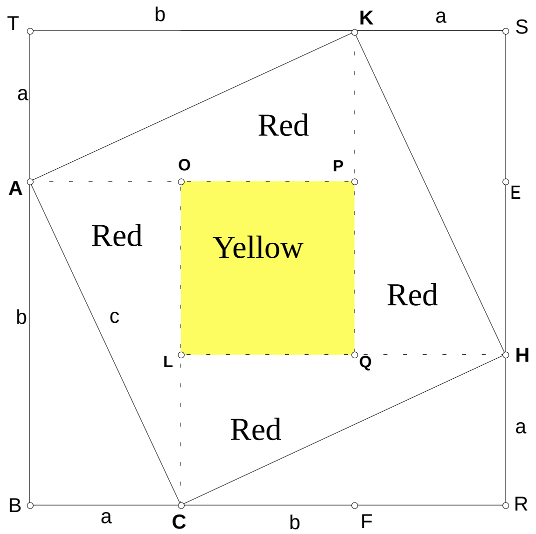
Web-page:

1. http://www.edp.ust.hk/math/history
2. http://www.cmi.hk/Teaching/
3. http://alepho.clarku.edu/~djoyce/mathhist
4. http://java.sun.com/applets/archive/beta/Pythagoras/
5. http://www.ies.co.jp/math/java/samples/pytha2.html (animation for the proof)
6. http://www.vjc.moe.edu.sg/dept/physics/applet/pytha/pythagoras.htm (animation for the proof)
7. http://www.sunsite.ubc.ca/DigitalMathArchive/Euclid/java/html/pythagoras.html (animation for the Euclid's proof)

Annex

The *Gou-gu* Theorem was a famous finding in ancient Chinese mathematics, particularly in geometry, and had a wide application. Far from years of San Guo (around 300AD), Zhao Shuang wrote his commentaries on *Zhou Bi Suan Jing* and the *Gou gu yuan tu zhu*. Discussion on the *Gou-gu* Theorem and problems related to *gou-gu* has been recorded in books like *Zhou Bi Suan Jing* and *Nine Chapters of Mathematical Art*. Zhao Shuang was in the years of around 3 to 4 Centuries. His works in mathematics was mainly kept in the commentaries of *Zhou Bi Suan Jiang*. One of them is the precious *Gou gu yuan tu zhu*. *Gou gu yuan tu zhu* was kept as one chapters of the present circulated *Zhou Bi Suan Jiang*. The whole writing is less than 500 hundred words but it included 21 statements about relations on right-angled triangles such as the *Gou-gu Theorem* and the *Yuan Tu*.

Zhao's proof was special. First, he used 4 identical right-angled triangles and formed the shape as the figure in the right. He then started to find out the area of the whole figure.



It is obvious that the figure *ACHK* is a square with side c and area c^2. On the other hand, the figure *ACHK* is composed of 4 right-angled triangles marked as "red" and a small square marked as "yellow".

The areas of the 4 triangles are 2ab and the area of the small square is $(b-a)^2$. Their sum is $2ab + (b-a)^2$ and after simplifying become $a^2 + b^2$. By comparing the 2 methods in finding the area of the square *ACHK*, the result follows:

$$a^2 + b^2 = c^2.$$

The proof presented by Zhao Shuang reflects the characteristics of proofs in ancient China. This included translating, combining, pasting and sticking the shapes and then using algebraic method to do geometric proof. This method integrates the method of geometry and algebra in problem solving. It is not only vigorous, but also provides intuitive impression. This special style is different from that in the Ancient West.

Translated from 《數學奇觀》 P.60